Vector Identity 7

$$\nabla \cdot (\psi \mathbf{a}) = \mathbf{a} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{a}$$

Proof

Let $f = \psi$ and let $\mathbf{A} = \mathbf{a}$.

$$\begin{aligned}
\nabla \cdot (f\mathbf{A}) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left[f\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j f\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \frac{\partial}{\partial x_i} A_j f \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \left(A_j \frac{\partial f}{\partial x_i} + \frac{\partial A_j}{\partial x_i} f\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) A_j \frac{\partial f}{\partial x_i} + \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \frac{\partial A_j}{\partial x_i} f \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) A_i \frac{\partial f}{\partial x_j} + f\left[\sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \frac{\partial A_j}{\partial x_i}\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i A_i\right) \cdot \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial f}{\partial x_j}\right) + f\left[\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right)\right] \\
&= \mathbf{A} \cdot \nabla f + f(\nabla \cdot \mathbf{A})
\end{aligned}$$