Vector Identity 9

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

Proof

Let $\mathbf{A} = \mathbf{a}$ and $\mathbf{B} = \mathbf{b}$.

$$\begin{aligned}
\mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i A_i\right)\times\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\times\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k B_k\right)\right] \\
&\quad + \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i B_i\right)\times\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\times\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k A_k\right)\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i A_i\right)\times\left[\sum_{j=1}^{3}\sum_{k=1}^{3}(\boldsymbol{\delta}_j\times\boldsymbol{\delta}_k)\frac{\partial}{\partial x_j}B_k\right] \\
&\quad + \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i B_i\right)\times\left[\sum_{j=1}^{3}\sum_{k=1}^{3}(\boldsymbol{\delta}_j\times\boldsymbol{\delta}_k)\frac{\partial}{\partial x_j}A_k\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i A_i\right)\times\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\varepsilon_{jkl}\frac{\partial B_k}{\partial x_j}\right) \\
&\quad + \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i B_i\right)\times\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\varepsilon_{jkl}\frac{\partial A_k}{\partial x_j}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_l)\varepsilon_{jkl}A_i\frac{\partial B_k}{\partial x_j} \\
&\quad + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_l)\varepsilon_{jkl}B_i\frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{mil}\varepsilon_{jkl}A_i\frac{\partial B_k}{\partial x_j} \\
&\quad + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{mil}\varepsilon_{jkl}B_i\frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m(\delta_{mj}\delta_{ik}-\delta_{mk}\delta_{ij})A_i\frac{\partial B_k}{\partial x_j} \\
&\quad + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m(\delta_{mj}\delta_{ik}-\delta_{mk}\delta_{ij})B_i\frac{\partial A_k}{\partial x_j}
\end{aligned}$$

Continue the simplification.

$$
\begin{aligned}
\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A}) &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\delta_{mj}\delta_{ik}A_i\frac{\partial B_k}{\partial x_j} - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\delta_{mk}\delta_{ij}A_i\frac{\partial B_k}{\partial x_j} \\
&\quad + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\delta_{mj}\delta_{ik}B_i\frac{\partial A_k}{\partial x_j} - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\delta_{mk}\delta_{ij}B_i\frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\delta_{ik}A_i\frac{\partial B_k}{\partial x_j} - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k\delta_{ij}A_i\frac{\partial B_k}{\partial x_j} \\
&\quad + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\delta_{ik}B_i\frac{\partial A_k}{\partial x_j} - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k\delta_{ij}B_i\frac{\partial A_k}{\partial x_j} \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_jA_k\frac{\partial B_k}{\partial x_j} - \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_kA_j\frac{\partial B_k}{\partial x_j} \\
&\quad + \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_jB_k\frac{\partial A_k}{\partial x_j} - \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_kB_j\frac{\partial A_k}{\partial x_j} \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\left(A_k\frac{\partial B_k}{\partial x_j}+B_k\frac{\partial A_k}{\partial x_j}\right) \\
&\quad - \sum_{j=1}^{3}A_j\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3}\boldsymbol{\delta}_kB_k\right) - \sum_{j=1}^{3}B_j\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3}\boldsymbol{\delta}_kA_k\right) \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}(A_kB_k) - \sum_{j=1}^{3}A_j\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3}\boldsymbol{\delta}_kB_k\right) - \sum_{j=1}^{3}B_j\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3}\boldsymbol{\delta}_kA_k\right) \\
&= \sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3}A_kB_k\right) - \sum_{j=1}^{3}A_j\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3}\boldsymbol{\delta}_kB_k\right) - \sum_{j=1}^{3}B_j\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3}\boldsymbol{\delta}_kA_k\right) \\
&= \nabla(\mathbf{A}\cdot\mathbf{B}) - (\mathbf{A}\cdot\nabla)\mathbf{B} - (\mathbf{B}\cdot\nabla)\mathbf{A}
\end{aligned}
$$

Therefore,

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}).$$