Problem 1.14

Two points

Consider two points located at $\mathbf{r}_1$ and $\mathbf{r}_2$, and separated by distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$. Find a time-dependent vector $\mathbf{A}(t)$ from the origin that is at $\mathbf{r}_1$ at time t_1 and at $\mathbf{r}_2$ at time $t_2 = t_1 + T$. Assume that $\mathbf{A}(t)$ moves uniformly along the straight line between the two points.

Solution

If $\mathbf{A}(t)$ moves uniformly along the straight line between $\mathbf{r}_1$ and $\mathbf{r}_2$, then $\mathbf{A}(t)$ must have a constant rate of change.

$$\mathbf{A}'(t) = \mathbf{C}$$

Integrate both sides with respect to t.

$$\mathbf{A}(t) = \mathbf{C}t + \mathbf{D}$$

Apply the two conditions, $\mathbf{A}(t_1) = \mathbf{r}_1$ and $\mathbf{A}(t_2) = \mathbf{r}_2$, to obtain two vector equations that can be solved for $\mathbf{C}$ and $\mathbf{D}$.

$$\mathbf{A}(t_1) = \mathbf{C}t_1 + \mathbf{D} = \mathbf{r}_1 \tag{1}$$

$$\mathbf{A}(t_2) = \mathbf{C}t_2 + \mathbf{D} = \mathbf{r}_2 \tag{2}$$

Subtract both sides of equation (1) from those of equation (2).

$$\mathbf{C}t_2 - \mathbf{C}t_1 = \mathbf{r}_2 - \mathbf{r}_1$$

Solve this equation for $\mathbf{C}$.

$$\mathbf{C} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1}$$

Now solve equation (1) for $\mathbf{D}$.

$$\begin{aligned}\mathbf{D} &= \mathbf{r}_1 - \mathbf{C}t_1 \\ &= \mathbf{r}_1 - \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1}t_1 \\ &= \frac{\mathbf{r}_1(t_2 - t_1) - (\mathbf{r}_2 - \mathbf{r}_1)t_1}{t_2 - t_1} \\ &= \frac{\mathbf{r}_1 t_2 - \mathbf{r}_2 t_1}{t_2 - t_1}\end{aligned}$$

Now that $\mathbf{C}$ and $\mathbf{D}$ are known, $\mathbf{A}(t)$ is as well.

$$\begin{aligned}\mathbf{A}(t) &= \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1}t + \frac{\mathbf{r}_1 t_2 - \mathbf{r}_2 t_1}{t_2 - t_1} \\ &= \frac{\mathbf{r}_1 t_2 - \mathbf{r}_1 t + \mathbf{r}_2 t - \mathbf{r}_2 t_1}{t_2 - t_1}\end{aligned}$$

Therefore,

$$\mathbf{A}(t) = \frac{t_2 - t}{t_2 - t_1}\mathbf{r}_1 + \frac{t - t_1}{t_2 - t_1}\mathbf{r}_2.$$