Problem 1.21

Particle with constant radial velocity

A particle moves in a plane with constant radial velocity $\dot{r} = 4$ m/s, starting from the origin. The angular velocity is constant and has magnitude $\dot{\theta} = 2$ rad/s. When the particle is 3 m from the origin, find the magnitude of (a) the velocity and (b) the acceleration.

Solution

Differentiate both sides of the radial and angular velocities to obtain the radial and angular accelerations.

$$\dot{r} = 4 \text{ m/s} \quad \rightarrow \quad \ddot{r} = 0$$
$$\dot{\theta} = 2 \text{ rad/s} \quad \rightarrow \quad \ddot{\theta} = 0$$

Integrate both sides of the radial velocity to obtain the radial position.

$$r(t) = 4t + C_1 \text{ m}$$

The fact that the particle starts from the origin means the initial condition is $r(0) = 0$, so $C_1 = 0$.

$$r(t) = 4t \text{ m}$$

The position vector in polar coordinates is

$$\mathbf{r} = r\hat{\mathbf{r}} = \{4t\hat{\mathbf{r}}\} \text{ m}.$$

Its magnitude is 3 meters when $4t = 3$, or

$$t = \frac{3}{4}.$$

The velocity and acceleration vectors in polar coordinates are

$$\mathbf{v}(t) = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \qquad \mathbf{a}(t) = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$$
$$\mathbf{v}(t) = \{4\hat{\mathbf{r}} + 4t(2)\hat{\boldsymbol{\theta}}\}\ \frac{\text{m}}{\text{s}} \qquad \mathbf{a}(t) = \{[0 - 4t(2)^2]\hat{\mathbf{r}} + [0 + 2(4)(2)]\hat{\boldsymbol{\theta}}\}\ \frac{\text{m}}{\text{s}^2}$$
$$\mathbf{v}(t) = \{4\hat{\mathbf{r}} + 8t\hat{\boldsymbol{\theta}}\}\ \frac{\text{m}}{\text{s}} \qquad \mathbf{a}(t) = \{-16t\hat{\mathbf{r}} + 16\hat{\boldsymbol{\theta}}\}\ \frac{\text{m}}{\text{s}^2}.$$

Evaluate the velocity and acceleration vectors at $t = 3/4$.

$$\mathbf{v}\left(\frac{3}{4}\right) = \left\{4\hat{\mathbf{r}} + 8\left(\frac{3}{4}\right)\hat{\boldsymbol{\theta}}\right\}\frac{\text{m}}{\text{s}} \qquad \mathbf{a}\left(\frac{3}{4}\right) = \left\{-16\left(\frac{3}{4}\right)\hat{\mathbf{r}} + 16\hat{\boldsymbol{\theta}}\right\}\frac{\text{m}}{\text{s}^2}$$
$$\mathbf{v}\left(\frac{3}{4}\right) = \left\{4\hat{\mathbf{r}} + 6\hat{\boldsymbol{\theta}}\right\}\frac{\text{m}}{\text{s}} \qquad \mathbf{a}\left(\frac{3}{4}\right) = \left\{-12\hat{\mathbf{r}} + 16\hat{\boldsymbol{\theta}}\right\}\frac{\text{m}}{\text{s}^2}$$

Therefore, when the particle is 3 meters from the origin the magnitudes of the velocity and acceleration vectors are

$$\left|\mathbf{v}\left(\frac{3}{4}\right)\right| = \sqrt{4^2 + 6^2}\ \frac{\text{m}}{\text{s}} \qquad \left|\mathbf{a}\left(\frac{3}{4}\right)\right| = \sqrt{(-12)^2 + 16^2}\ \frac{\text{m}}{\text{s}^2}$$
$$\left|\mathbf{v}\left(\frac{3}{4}\right)\right| = \sqrt{52}\ \frac{\text{m}}{\text{s}} \qquad \left|\mathbf{a}\left(\frac{3}{4}\right)\right| = 20\ \frac{\text{m}}{\text{s}^2}.$$