Problem 1.5

Perpendicular vectors
Show that if $|\mathbf{A} - \mathbf{B}| = |\mathbf{A} + \mathbf{B}|$, then $\mathbf{A}$ and $\mathbf{B}$ are perpendicular.

Solution

The strategy for this problem is to show that $\mathbf{A} \cdot \mathbf{B} = 0$. From this we can conclude that the vectors are perpendicular. Start off with the given equation.

$$|\mathbf{A} - \mathbf{B}| = |\mathbf{A} + \mathbf{B}|$$

The magnitude of a vector is the square root of the sum of its components squared. Thus, if $\mathbf{A}$ and $\mathbf{B}$ are n-dimensional, then

$$\sqrt{\sum_{i=1}^{n}(a_i - b_i)^2} = \sqrt{\sum_{i=1}^{n}(a_i + b_i)^2}.$$

Square both sides of the equation to eliminate the square roots.

$$\sum_{i=1}^{n}(a_i - b_i)^2 = \sum_{i=1}^{n}(a_i + b_i)^2$$

Expand each of the squared terms.

$$\sum_{i=1}^{n}(a_i^2 - 2a_ib_i + b_i^2) = \sum_{i=1}^{n}(a_i^2 + 2a_ib_i + b_i^2)$$

Split up the series and cancel common terms on both sides.

$$\cancel{\sum_{i=1}^{n} a_i^2} - 2\sum_{i=1}^{n} a_ib_i + \cancel{\sum_{i=1}^{n} b_i^2} = \cancel{\sum_{i=1}^{n} a_i^2} + 2\sum_{i=1}^{n} a_ib_i + \cancel{\sum_{i=1}^{n} b_i^2}$$

All that remains are the cross terms.

$$-2\sum_{i=1}^{n} a_ib_i = 2\sum_{i=1}^{n} a_ib_i$$

$$-4\sum_{i=1}^{n} a_ib_i = 0$$

$$\sum_{i=1}^{n} a_ib_i = 0$$

The sum of the products of the vectors' respective components is the definition of the dot product.

$$\mathbf{A} \cdot \mathbf{B} = 0$$

The left side can be written in terms of the angle between the vectors.

$$|\mathbf{A}||\mathbf{B}|\cos\theta = 0$$

The magnitudes are nonzero, so the cosine of the angle between the vectors must be 0.

$$\cos\theta = 0$$

Taking the inverse cosine of both sides gives the angle between the vectors.

$$\theta = \frac{\pi}{2}$$

Therefore, **A** and **B** are perpendicular.