Problem 1.7

*Law of sines**
Prove the law of sines using the cross product. It should only take a couple of lines.

Solution

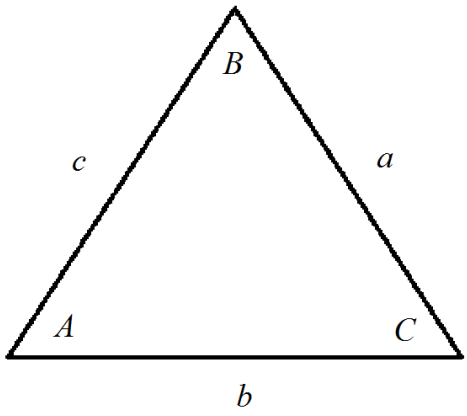

Figure 1: Schematic of a triangle.

We have to prove the law of sines, which states that the following must hold for a triangle.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The magnitude of a cross product is defined to be the product of the vectors' magnitudes times the sine of the angle between them.

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}| \sin\theta$$

If we treat each side of the triangle in the figure as a vector, then we can take the cross product of the different sides.

$$|\overrightarrow{AC} \times \overrightarrow{AB}| = |\overrightarrow{AC}||\overrightarrow{AB}| \sin A = bc \sin A$$
$$|\overrightarrow{BC} \times \overrightarrow{BA}| = |\overrightarrow{BC}||\overrightarrow{BA}| \sin C = ac \sin B$$
$$|\overrightarrow{AC} \times \overrightarrow{BC}| = |\overrightarrow{AC}||\overrightarrow{BC}| \sin C = ab \sin C$$

The magnitude of the cross product represents the area of the parallelogram created from the two vectors. Since $|\overrightarrow{AC}|$, $|\overrightarrow{BC}|$, and $|\overrightarrow{BA}|$ are the sides of the same triangle, the three expressions above must be equal.

$$bc \sin A = ac \sin B = ab \sin C$$

If we divide each side by abc, then we get the law of sines.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$