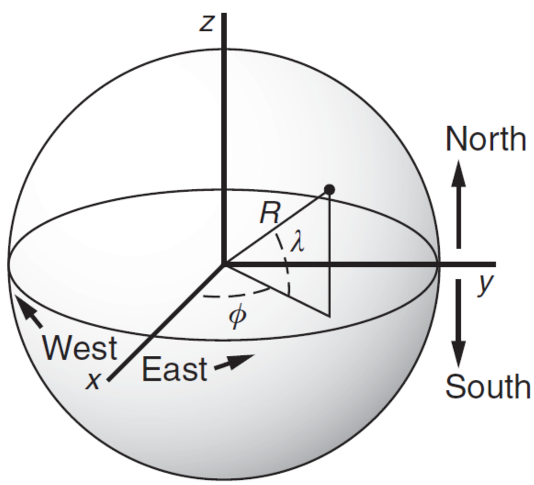
Problem 1.15

Great circle
The shortest distance between two points on the Earth (considered to be a perfect sphere of radius R) is the distance along a great circle—the arc of a circle formed where a plane passing through the two points and the center of the Earth intersects the Earth's surface.

The position of a point on the Earth is specified by the point's longitude ϕ and latitude λ. Longitude is the angle between the between the meridian (a line from pole to pole) passing through the point and the "prime" meridian passing through Greenwich U.K. Longitude is taken to be positive to the east and negative to the west. Latitude is the angle from the Equator along the point's meridian, taken positive to the north.

Let the vectors from the center of the Earth to the two points be $\mathbf{r}_1$ and $\mathbf{r}_2$. The cosine of the angle θ between them can be found from their dot product, so that the great circle distance between the points is $R\theta$.

Find an expression for θ in terms of the coordinates of the two points. Use a coordinate system with the x axis in the equatorial plane and passing through the prime meridian; let the z axis be on the polar axis, positive toward the north pole, as shown in the sketch.



Solution

For the point on the sphere in the figure above, we observe the following relationships.

$$\begin{aligned} x &= (R\cos\lambda)\cos\phi \\ y &= (R\cos\lambda)\sin\phi \\ z &= R\sin\lambda \end{aligned}$$

The position vectors of two points on the sphere are then

$$\begin{aligned} \mathbf{r}_1 &= \langle x_1, y_1, z_1\rangle \\ &= \langle R\cos\lambda_1\cos\phi_1, R\cos\lambda_1\sin\phi_1, R\sin\lambda_1\rangle \\ \mathbf{r}_2 &= \langle x_2, y_2, z_2\rangle \\ &= \langle R\cos\lambda_2\cos\phi_2, R\cos\lambda_2\sin\phi_2, R\sin\lambda_2\rangle. \end{aligned}$$

The dot product of $\mathbf{r}_1$ and $\mathbf{r}_2$ is defined as

$$\mathbf{r}_1\cdot\mathbf{r}_2 = |\mathbf{r}_1||\mathbf{r}_2|\cos\theta,$$

where θ is the angle between them. Note that $|\mathbf{r}_1|$ and $|\mathbf{r}_2|$ are both R, the distance from the Earth's center to the Earth's surface. Compute $\mathbf{r}_1 \cdot \mathbf{r}_2$ now.

$$\begin{aligned}(R\cos\lambda_1\cos\phi_1)(R\cos\lambda_2\cos\phi_2) + (R\cos\lambda_1\sin\phi_1)(R\cos\lambda_2\sin\phi_2) + (R\sin\lambda_1)(R\sin\lambda_2) &= R^2\cos\theta \\ R^2\cos\lambda_1\cos\lambda_2\cos\phi_1\cos\phi_2 + R^2\cos\lambda_1\cos\lambda_2\sin\phi_1\sin\phi_2 + R^2\sin\lambda_1\sin\lambda_2 &= R^2\cos\theta\end{aligned}$$

Divide both sides by R^2 and factor $\cos\lambda_1\cos\lambda_2$.

$$\cos\lambda_1\cos\lambda_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2) + \sin\lambda_1\sin\lambda_2 = \cos\theta$$

Use the angle subtraction formula for cosine.

$$\cos\lambda_1\cos\lambda_2\cos(\phi_1 - \phi_2) + \sin\lambda_1\sin\lambda_2 = \cos\theta$$

Therefore, taking the inverse cosine of both sides,

$$\theta = \cos^{-1}[\cos\lambda_1\cos\lambda_2\cos(\phi_1 - \phi_2) + \sin\lambda_1\sin\lambda_2].$$