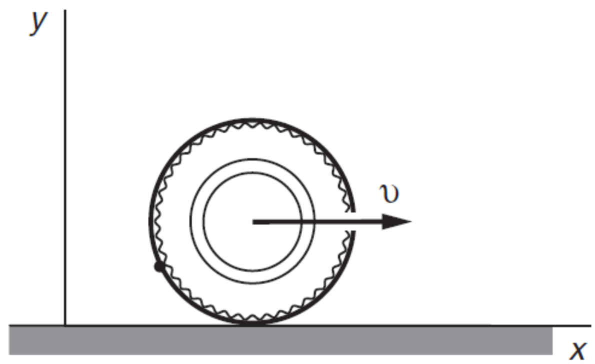
Problem 1.24

Rolling tire

A tire of radius R rolls in a straight line without slipping. Its center moves with constant speed V. A small pebble lodged in the tread of the tire touches the road at $t = 0$. Find the pebble's position, velocity, and acceleration as functions of time.



[TYPO: It should be V in the figure, not v.]

Solution

In order to find the position vector of the pebble with respect to the origin $\mathbf{r}$, we will find the position of the tire's center with respect to the origin $\mathbf{r}_c$ and the position of the pebble with respect to the tire's center $\mathbf{r}_p$ and then add them vectorially.

$$\mathbf{r} = \mathbf{r}_c + \mathbf{r}_p$$

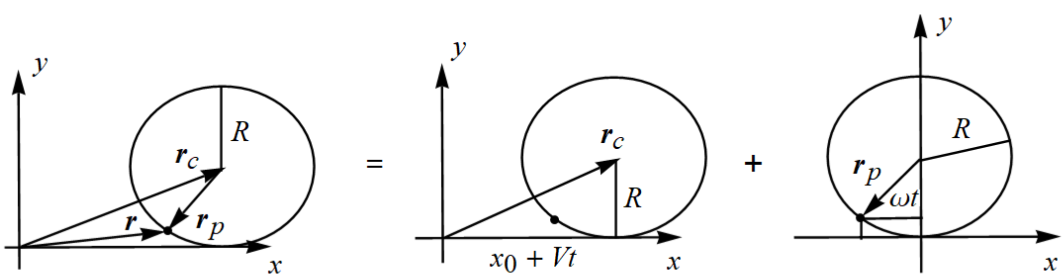


Figure 1: This is an illustration of the process for finding $\mathbf{r}$.

The position of the tire's center is found using the kinematic formula,

$$x = x_0 + v_0 t + \frac{1}{2}at^2.$$

Since the tire is moving at constant speed V, $v_0 = V$ and $a = 0$. Also, the center is always a distance of R above the ground.

$$\mathbf{r}_c = (x_0 + Vt)\hat{\mathbf{x}} + R\hat{\mathbf{y}}$$

t seconds after the tire is set in motion, an angle of ωt is swept out as shown in Figure 1, where ω is the angular speed that the tire rotates. The position of the pebble lies to the left of the origin, so the x-component will be negative. The y-component is the vertical distance from the origin to the pebble.

$$\mathbf{r}_p = -R\sin(\omega t)\hat{\mathbf{x}} + (R - R\cos\omega t)\hat{\mathbf{y}}$$

Here ω will be written in terms of the known quantities, R and V. Because $V = R\omega$, $\omega = V/R$. So

$$\mathbf{r}_p = -R\sin\left(\frac{Vt}{R}\right)\hat{\mathbf{x}} + \left(R - R\cos\frac{Vt}{R}\right)\hat{\mathbf{y}}.$$

With $\mathbf{r}_c$ and $\mathbf{r}_p$ in hand, $\mathbf{r}$ is known.

$$\begin{aligned}\mathbf{r} &= \mathbf{r}_c + \mathbf{r}_p \\ &= \left(x_0 + Vt - R\sin\frac{Vt}{R}\right)\hat{\mathbf{x}} + \left(2R - R\cos\frac{Vt}{R}\right)\hat{\mathbf{y}}\end{aligned}$$

Therefore,

$$\boxed{\mathbf{r}(t) = R\left[\left(\frac{x_0}{R} + \frac{Vt}{R} - \sin\frac{Vt}{R}\right)\hat{\mathbf{x}} + \left(2 - \cos\frac{Vt}{R}\right)\hat{\mathbf{y}}\right],}$$

where x_0 is the initial position of the pebble along the x-axis. Notice that each of the fractions is dimensionless. To find the velocity, take the derivative of the position vector with respect to time.

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = R\left[\left(\frac{V}{R} - \frac{V}{R}\cos\frac{Vt}{R}\right)\hat{\mathbf{x}} + \left(\frac{V}{R}\sin\frac{Vt}{R}\right)\hat{\mathbf{y}}\right]$$

Therefore,

$$\boxed{\mathbf{v}(t) = V\left[\left(1 - \cos\frac{Vt}{R}\right)\hat{\mathbf{x}} + \sin\left(\frac{Vt}{R}\right)\hat{\mathbf{y}}\right].}$$

To find the acceleration, take the derivative of the velocity vector with respect to time.

$$\mathbf{a}(t) = \dot{\mathbf{v}}(t) = V\left[\left(\frac{V}{R}\sin\frac{Vt}{R}\right)\hat{\mathbf{x}} + \frac{V}{R}\cos\left(\frac{Vt}{R}\right)\hat{\mathbf{y}}\right]$$

Therefore,

$$\boxed{\mathbf{a}(t) = \frac{V^2}{R}\left[\sin\left(\frac{Vt}{R}\right)\hat{\mathbf{x}} + \cos\left(\frac{Vt}{R}\right)\hat{\mathbf{y}}\right].}$$

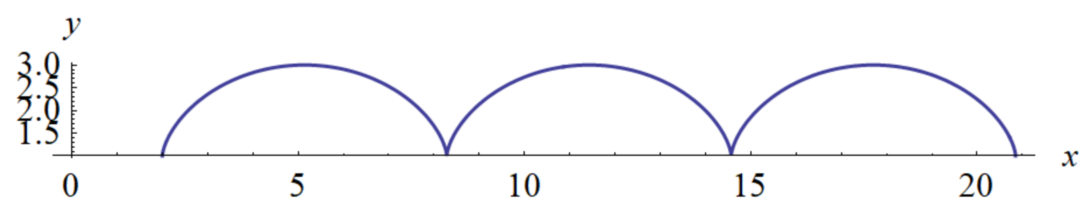


Figure 2: This is a plot of the pebble's path (known as a cycloid) for $0 \le t \le 6\pi$, $x_0 = 2$, $V = 1$, and $R = 1$.