Exercise 28

Suppose a, b, c, and d are constants such that a is not zero and the system below is consistent for all possible values of f and g. What can you say about the numbers a, b, c, and d? Justify your answer.

$$\begin{aligned} ax_1 + bx_2 &= f \\ cx_1 + dx_2 &= g \end{aligned}$$

Solution

Write the augmented matrix corresponding to this system of equations.

$$\left[\begin{array}{cc|c} a & b & f \\ c & d & g \end{array}\right]$$

To make the top left entry 1, divide the first row by a.

$$\left[\begin{array}{cc|c} 1 & \frac{b}{a} & \frac{f}{a} \\ c & d & g \end{array}\right]$$

To make the bottom left entry 0, multiply the first row by $-c$ and add it to the second row.

$$\left[\begin{array}{cc|c} 1 & \frac{b}{a} & \frac{f}{a} \\ 0 & d - \frac{bc}{a} & g - \frac{cf}{a} \end{array}\right]$$

Simplify the matrix.

$$\left[\begin{array}{cc|c} 1 & \frac{b}{a} & \frac{f}{a} \\ 0 & \frac{ad-bc}{a} & \frac{ag-cf}{a} \end{array}\right] \tag{1}$$

Divide the second row by $(ad - bc)/a$, assuming $ad - bc \neq 0$.

$$\left[\begin{array}{cc|c} 1 & \frac{b}{a} & \frac{f}{a} \\ 0 & 1 & \frac{ag-cf}{ad-bc} \end{array}\right]$$

Multiply the second row by $-b/a$ and add it to the first row.

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{f}{a} - \frac{b}{a}\frac{ag-cf}{ad-bc} \\ 0 & 1 & \frac{ag-cf}{ad-bc} \end{array}\right]$$

Simplify the matrix.

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{df-bg}{ad-bc} \\ 0 & 1 & \frac{ag-cf}{ad-bc} \end{array}\right]$$

The solution is then

$$x_1 = \frac{df - bg}{ad - bc} \quad \text{and} \quad x_2 = \frac{ag - cf}{ad - bc}$$

if $ad - bc \neq 0$. If $ad - bc = 0$, then equation (1) becomes

$$\left[\begin{array}{cc|c} 1 & \frac{b}{a} & \frac{f}{a} \\ 0 & 0 & \frac{ag-cf}{a} \end{array}\right].$$

The last row implies that $0x_1 + 0x_2 = (ag - cf)/a$. No values of x_1 and x_2 can make this a true statement, since the right side is nonzero. Therefore, there's no solution if $ad - bc = 0$.