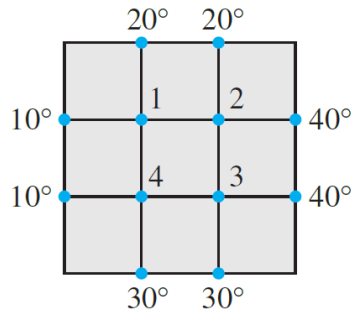


Exercise 34

An important concern in the study of heat transfer is to determine the steady-state temperature distribution of a thin plate when the temperature around the boundary is known. Assume the plate shown in the figure represents a cross section of a metal beam, with negligible heat flow in the direction perpendicular to the plate. Let T_1, \dots, T_4 denote the temperatures at the four interior nodes of the mesh in the figure. The temperature at a node is approximately equal to the average of the four nearest nodes—to the left, above, to the right, and below.² For instance,

$$T_1 = (10 + 20 + T_2 + T_4)/4 \quad \text{or} \quad 4T_1 - T_2 - T_4 = 30$$



Solve the system of equations from Exercise 33. [*Hint:* To speed up the calculations, interchange rows 1 and 4 before starting “replace” operations.]

Solution

The temperature at each node is the average of the temperatures around it.

$$\text{Node 1:} \quad T_1 = \frac{10 + 20 + T_2 + T_4}{4} \quad \rightarrow \quad 4T_1 - T_2 - T_4 = 30$$

$$\text{Node 2:} \quad T_2 = \frac{20 + 40 + T_1 + T_3}{4} \quad \rightarrow \quad 4T_2 - T_1 - T_3 = 60$$

$$\text{Node 3:} \quad T_3 = \frac{30 + 40 + T_2 + T_4}{4} \quad \rightarrow \quad 4T_3 - T_2 - T_4 = 70$$

$$\text{Node 4:} \quad T_4 = \frac{10 + 30 + T_1 + T_3}{4} \quad \rightarrow \quad 4T_4 - T_1 - T_3 = 40$$

Write the augmented matrix corresponding to this system of equations.

$$\left[\begin{array}{cccc|c} 4 & -1 & 0 & -1 & 30 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ -1 & 0 & -1 & 4 & 40 \end{array} \right]$$

The aim is to write this matrix in triangular form. Start by switching the fourth row with the first row.

$$\left[\begin{array}{cccc|c} -1 & 0 & -1 & 4 & 40 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ 4 & -1 & 0 & -1 & 30 \end{array} \right]$$

²See Frank M. White, *Heat and Mass Transfer* (Reading, MA: Addison-Wesley Publishing, 1991), pp. 145–149.

Multiply the first row by -1 to make the top left entry 1.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -4 & -40 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ 4 & -1 & 0 & -1 & 30 \end{array} \right]$$

Add the first row to the second row.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -4 & -40 \\ 0 & 4 & 0 & -4 & 20 \\ 0 & -1 & 4 & -1 & 70 \\ 4 & -1 & 0 & -1 & 30 \end{array} \right]$$

Divide the second row by 4.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & -1 & 4 & -1 & 70 \\ 4 & -1 & 0 & -1 & 30 \end{array} \right]$$

Multiply the first row by -4 and add it to the fourth row.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & -1 & 4 & -1 & 70 \\ 0 & -1 & -4 & 15 & 190 \end{array} \right]$$

Add the second row to the third row.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 4 & -2 & 75 \\ 0 & -1 & -4 & 15 & 190 \end{array} \right]$$

Add the second row to the fourth row.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 4 & -2 & 75 \\ 0 & 0 & -4 & 14 & 195 \end{array} \right]$$

Add the third row to the fourth row.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 4 & -2 & 75 \\ 0 & 0 & 0 & 12 & 270 \end{array} \right]$$

Divide the fourth row by 12.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 4 & -2 & 75 \\ 0 & 0 & 0 & 1 & 22.5 \end{array} \right]$$

Multiply the fourth row by 2 and add it to the third row.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 4 & 0 & 120 \\ 0 & 0 & 0 & 1 & 22.5 \end{array} \right]$$

Divide the third row by 4.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 1 & 0 & 30 \\ 0 & 0 & 0 & 1 & 22.5 \end{array} \right]$$

Add the fourth row to the second row.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & 0 & 27.5 \\ 0 & 0 & 1 & 0 & 30 \\ 0 & 0 & 0 & 1 & 22.5 \end{array} \right]$$

Multiply the fourth row by 4 and add it to the first row.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 50 \\ 0 & 1 & 0 & 0 & 27.5 \\ 0 & 0 & 1 & 0 & 30 \\ 0 & 0 & 0 & 1 & 22.5 \end{array} \right]$$

Multiply the third row by -1 and add it to the first row.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & 27.5 \\ 0 & 0 & 1 & 0 & 30 \\ 0 & 0 & 0 & 1 & 22.5 \end{array} \right]$$

Therefore, the estimated temperatures inside the plate are

$$T_1 = 20^\circ \quad \text{and} \quad T_2 = 27.5^\circ \quad \text{and} \quad T_3 = 30^\circ \quad T_4 = 22.5^\circ.$$