Exercise 13

In Exercises 13 to 19, use set theoretic or vector notation or both to describe the points that lie in the given configurations.

The plane spanned by $\mathbf{v}_1 = (2, 7, 0)$ and $\mathbf{v}_2 = (0, 2, 7)$

Solution

These two vectors are linearly independent because one is not a constant multiple of the other. That means an entire plane is spanned by taking a linear combination of the two.

$$\begin{aligned} C_1\mathbf{v}_1 + C_2\mathbf{v}_2 &= C_1(2, 7, 0) + C_2(0, 2, 7) \\ &= (2C_1, 7C_1, 0) + (0, 2C_2, 7C_2) \\ &= (2C_1, 7C_1 + 2C_2, 7C_2) \end{aligned}$$

This plane is two-dimensional because there are two arbitrary constants, C_1 and C_2. The points in this plane are described in set notation by

$$\{(2C_1, 7C_1 + 2C_2, 7C_2),\ C_1 \in \mathbb{R},\ C_2 \in \mathbb{R}\}.$$

Any plane can be described by a vector perpendicular to it. For the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ in particular, a perpendicular vector can be obtained by taking the cross product.

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 2 & 7 & 0 \\ 0 & 2 & 7 \end{vmatrix} = 49\hat{\mathbf{x}} - 14\hat{\mathbf{y}} + 4\hat{\mathbf{z}}$$

Because this vector is normal to every vector lying in the plane, the dot product of these two is zero. The equation for a plane is obtained from this fact.

$$(\mathbf{v}_1 \times \mathbf{v}_2) \cdot (\mathbf{r} - \mathbf{v}_1) = 0$$

$$(49, -14, 4) \cdot [(x, y, z) - (2, 7, 0)] = 0$$

$$(49, -14, 4) \cdot (x - 2, y - 7, z) = 0$$

$$49(x - 2) - 14(y - 7) + 4z = 0$$

Another way to describe the points in the plane using set notation is

$$\{(x, y, z) \mid 49(x - 2) - 14(y - 7) + 4z = 0\}.$$