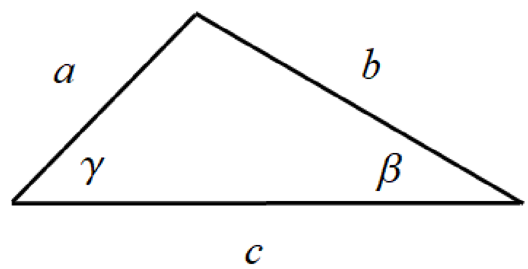
Exercise 32

Prove the statements in Exercises 32 to 34.

The line segment joining the midpoints of two sides of a triangle is parallel to and has half the length of the third side.

Solution

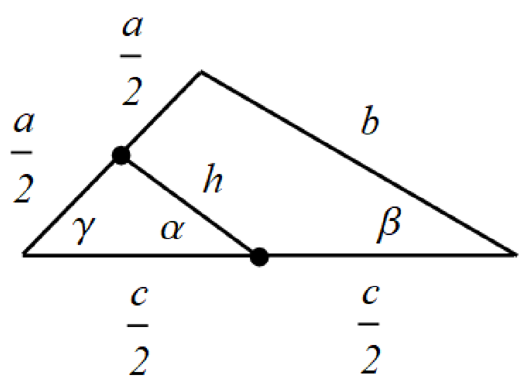
Consider a triangle with sides a, b, and c.



According to the law of cosines,

$$b^2 = a^2 + c^2 - 2ac\cos\gamma \quad \rightarrow \quad \cos\gamma = \frac{b^2 - a^2 - c^2}{-2ac}.$$

Draw the line segment joining the midpoints of two sides and let its length be h.



The aim is to show that $h = b/2$ and that $\alpha = \beta$. Use the law of cosines again for the smaller triangle.

$$\begin{aligned} h^2 &= \left(\frac{a}{2}\right)^2 + \left(\frac{c}{2}\right)^2 - 2\left(\frac{a}{2}\right)\left(\frac{c}{2}\right)\cos\gamma \\ &= \frac{a^2}{4} + \frac{c^2}{4} - 2\left(\frac{a}{2}\right)\left(\frac{c}{2}\right)\left(\frac{b^2 - a^2 - c^2}{-2ac}\right) \\ &= \frac{a^2}{4} + \frac{c^2}{4} - \frac{2ac}{4}\left(\frac{b^2 - a^2 - c^2}{-2ac}\right) \\ &= \frac{a^2}{4} + \frac{c^2}{4} + \frac{b^2 - a^2 - c^2}{4} \\ &= \frac{b^2}{4} \end{aligned}$$

Taking the square root of both sides yields $h = b/2$. Use the law of cosines again to obtain formulas involving α and β.

$$a^2 = b^2 + c^2 - 2bc\cos\beta$$
$$\left(\frac{a}{2}\right)^2 = h^2 + \left(\frac{c}{2}\right)^2 - 2h\left(\frac{c}{2}\right)\cos\alpha$$

Solve these two equations for the cosines.

$$\cos\beta = \frac{a^2 - b^2 - c^2}{-2bc}$$

The second equation becomes

$$\frac{a^2}{4} = \frac{b^2}{4} + \frac{c^2}{4} - 2\frac{b}{2}\left(\frac{c}{2}\right)\cos\alpha$$
$$\frac{a^2 - b^2 - c^2}{4} = -\frac{bc}{2}\cos\alpha$$
$$\frac{a^2 - b^2 - c^2}{-2bc} = \cos\alpha,$$

which means

$$\cos\alpha = \cos\beta$$
$$\alpha = \beta + 2n\pi, \quad n = 0, \pm1, \pm2, \ldots$$

Since both α and β are between 0 and 2π, $n = 0$.

$$\alpha = \beta$$

Therefore, the line segment joining the midpoints of two sides of a triangle is parallel to and has half the length of the third side.