Exercise 24

Let $\mathbf{b} = (3, 1, 1)$ and P be the plane through the origin given by $x + y + 2z = 0$.

(a) Find an orthogonal basis for P. That is, find two nonzero orthogonal vectors $\mathbf{v}_1, \mathbf{v}_2 \in P$.

(b) Find the orthogonal projection of $\mathbf{b}$ onto P. That is, find $\text{Proj}_{\mathbf{v}_1} \mathbf{b} + \text{Proj}_{\mathbf{v}_2} \mathbf{b}$.

Solution

Part (a)

The equation of the plane can be written as

$$x + y + 2z = (1, 1, 2) \cdot (x, y, z) = 0.$$

The vector $(1, 1, 2)$ is orthogonal to the plane at every point. Choose values for x, y, and z that satisfy the equation, for example, $x = 1$, $y = 1$, and $z = -1$. Let this be $\mathbf{v}_1$.

$$\mathbf{v}_1 = (1, 1, -1)$$

To get a second vector in the plane orthogonal to the first, take the cross product of $(1, 1, 2)$ and $\mathbf{v}_1$.

$$\mathbf{v}_2 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 2 \\ 1 & 1 & -1 \end{vmatrix} = -3\hat{\mathbf{x}} + 3\hat{\mathbf{y}} = (-3, 3, 0)$$

Part (b)

$$\begin{aligned}
\text{Proj}_{\mathbf{v}_1} \mathbf{b} + \text{Proj}_{\mathbf{v}_2} \mathbf{b} &= (\mathbf{b} \cdot \hat{\mathbf{v}}_1)\hat{\mathbf{v}}_1 + (\mathbf{b} \cdot \hat{\mathbf{v}}_2)\hat{\mathbf{v}}_2 \\
&= \frac{\mathbf{b} \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v_1} + \frac{\mathbf{b} \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \\
&= \frac{(3, 1, 1) \cdot (1, 1, -1)}{1^2 + 1^2 + (-1)^2}(1, 1, -1) + \frac{(3, 1, 1) \cdot (-3, 3, 0)}{(-3)^2 + 3^2}(-3, 3, 0) \\
&= \frac{(3)(1) + (1)(1) + (1)(-1)}{3}(1, 1, -1) + \frac{(3)(-3) + (1)(3) + (1)(0)}{18}(-3, 3, 0) \\
&= \frac{3}{3}(1, 1, -1) + \frac{-6}{18}(-3, 3, 0) \\
&= (1, 1, -1) + (1, -1, 0) \\
&= (2, 0, -1)
\end{aligned}$$