Exercise 24

(a) Prove, without recourse to geometry, that

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u}) = \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) &= -\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) \\ &= -\mathbf{w}\cdot(\mathbf{v}\times\mathbf{u}) = -\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w}).\end{aligned}$$

(b) Use part (a) and Exercise 23(a) to prove that

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{u}'\times\mathbf{v}') &= (\mathbf{u}\cdot\mathbf{u}')(\mathbf{v}\cdot\mathbf{v}') - (\mathbf{u}\cdot\mathbf{v}')(\mathbf{u}'\cdot\mathbf{v}) \\ &= \begin{vmatrix}\mathbf{u}\cdot\mathbf{u}' & \mathbf{u}\cdot\mathbf{v}' \\ \mathbf{u}'\cdot\mathbf{v} & \mathbf{v}\cdot\mathbf{v}'\end{vmatrix}.\end{aligned}$$

Solution

Part (a)

Let $\mathbf{u} = (u_x, u_y, u_z)$ and $\mathbf{v} = (v_x, v_y, v_z)$ and $\mathbf{w} = (w_x, w_y, w_z)$.

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= (u_x, u_y, u_z)\cdot\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ v_x & v_y & v_z \\ w_x & w_y & w_z\end{vmatrix} \\ &= (u_x, u_y, u_z)\cdot[(v_yw_z - v_zw_y)\hat{\mathbf{x}} - (v_xw_z - v_zw_x)\hat{\mathbf{y}} + (v_xw_y - v_yw_x)\hat{\mathbf{z}}] \\ &= u_x(v_yw_z - v_zw_y) - u_y(v_xw_z - v_zw_x) + u_z(v_xw_y - v_yw_x) \\ &= u_xv_yw_z - u_xv_zw_y - u_yv_xw_z + u_yv_zw_x + u_zv_xw_y - u_zv_yw_x\end{aligned}$$

$$\begin{aligned}\mathbf{v}\cdot(\mathbf{w}\times\mathbf{u}) &= (v_x, v_y, v_z)\cdot\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ w_x & w_y & w_z \\ u_x & u_y & u_z\end{vmatrix} \\ &= (v_x, v_y, v_z)\cdot[(w_yu_z - w_zu_y)\hat{\mathbf{x}} - (w_xu_z - w_zu_x)\hat{\mathbf{y}} + (w_xu_y - w_yu_x)\hat{\mathbf{z}}] \\ &= v_x(w_yu_z - w_zu_y) - v_y(w_xu_z - w_zu_x) + v_z(w_xu_y - w_yu_x) \\ &= u_zv_xw_y - u_yv_xw_z - u_zv_yw_x + u_xv_yw_z + u_yv_zw_x - u_xv_zw_y \\ &= u_xv_yw_z - u_xv_zw_y - u_yv_xw_z + u_yv_zw_x + u_zv_xw_y - u_zv_yw_x\end{aligned}$$

$$\begin{aligned}\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) &= (w_x, w_y, w_z)\cdot\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ u_x & u_y & u_z \\ v_x & v_y & v_z\end{vmatrix} \\ &= (w_x, w_y, w_z)\cdot[(u_yv_z - u_zv_y)\hat{\mathbf{x}} - (u_xv_z - u_zv_x)\hat{\mathbf{y}} + (u_xv_y - u_yv_x)\hat{\mathbf{z}}] \\ &= w_x(u_yv_z - u_zv_y) - w_y(u_xv_z - u_zv_x) + w_z(u_xv_y - u_yv_x) \\ &= u_yv_zw_x - u_zv_yw_x - u_xv_zw_y + u_zv_xw_y + u_xv_yw_z - u_yv_xw_z \\ &= u_xv_yw_z - u_xv_zw_y - u_yv_xw_z + u_yv_zw_x + u_zv_xw_y - u_zv_yw_x\end{aligned}$$

$$\begin{aligned}
\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) &= (u_x,u_y,u_z)\cdot\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ w_x & w_y & w_z\\ v_x & v_y & v_z\end{vmatrix}\\
&= (u_x,u_y,u_z)\cdot[(w_yv_z-w_zv_y)\hat{\mathbf{x}}-(w_xv_z-w_zv_x)\hat{\mathbf{y}}+(w_xv_y-w_yv_x)\hat{\mathbf{z}}]\\
&= u_x(w_yv_z-w_zv_y)-u_y(w_xv_z-w_zv_x)+u_z(w_xv_y-w_yv_x)\\
&= u_xv_zw_y-u_xv_yw_z-u_yv_zw_x+u_yv_xw_z+u_zv_yw_x-u_zv_xw_y\\
&= -(u_xv_yw_z-u_xv_zw_y-u_yv_xw_z+u_yv_zw_x+u_zv_xw_y-u_zv_yw_x)
\end{aligned}$$

$$\begin{aligned}
\mathbf{w}\cdot(\mathbf{v}\times\mathbf{u}) &= (w_x,w_y,w_z)\cdot\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ v_x & v_y & v_z\\ u_x & u_y & u_z\end{vmatrix}\\
&= (w_x,w_y,w_z)\cdot[(v_yu_z-v_zu_y)\hat{\mathbf{x}}-(v_xu_z-v_zu_x)\hat{\mathbf{y}}+(v_xu_y-v_yu_x)\hat{\mathbf{z}}]\\
&= w_x(v_yu_z-v_zu_y)-w_y(v_xu_z-v_zu_x)+w_z(v_xu_y-v_yu_x)\\
&= u_zv_yw_x-u_yv_zw_x-u_zv_xw_y+u_xv_zw_y+u_yv_xw_z-u_xv_yw_z\\
&= -(u_xv_yw_z-u_xv_zw_y-u_yv_xw_z+u_yv_zw_x+u_zv_xw_y-u_zv_yw_x)
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w}) &= (v_x,v_y,v_z)\cdot\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ u_x & u_y & u_z\\ w_x & w_y & w_z\end{vmatrix}\\
&= (v_x,v_y,v_z)\cdot[(u_yw_z-u_zw_y)\hat{\mathbf{x}}-(u_xw_z-u_zw_x)\hat{\mathbf{y}}+(u_xw_y-u_yw_x)\hat{\mathbf{z}}]\\
&= v_x(u_yw_z-u_zw_y)-v_y(u_xw_z-u_zw_x)+v_z(u_xw_y-u_yw_x)\\
&= u_yv_xw_z-u_zv_xw_y-u_xv_yw_z+u_zv_yw_x+u_xv_zw_y-u_yv_zw_x\\
&= -(u_xv_yw_z-u_xv_zw_y-u_yv_xw_z+u_yv_zw_x+u_zv_xw_y-u_zv_yw_x)
\end{aligned}$$

Therefore, $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})=\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})=-\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v})=-\mathbf{w}\cdot(\mathbf{v}\times\mathbf{u})=-\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w})$.

Part (b)

Treat $(\mathbf{u}'\times\mathbf{v}')$ as $\mathbf{w}$, use the fact that the dot product is commutative, use the result from part (a), and then use the second result from part (a) of Exercise 23.

$$\begin{aligned}
(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{u}'\times\mathbf{v}') &= (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}\\
&= \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})\\
&= \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})\\
&= \mathbf{u}\cdot[\mathbf{v}\times(\mathbf{u}'\times\mathbf{v}')]\\
&= \mathbf{u}\cdot[(\mathbf{v}\cdot\mathbf{v}')\mathbf{u}'-(\mathbf{v}\cdot\mathbf{u}')\mathbf{v}']\\
&= \mathbf{u}\cdot[\mathbf{u}'(\mathbf{v}\cdot\mathbf{v}')-\mathbf{v}'(\mathbf{u}'\cdot\mathbf{v})]\\
&= (\mathbf{u}\cdot\mathbf{u}')(\mathbf{v}\cdot\mathbf{v}')-(\mathbf{u}\cdot\mathbf{v}')(\mathbf{u}'\cdot\mathbf{v})\\
&= \begin{vmatrix}\mathbf{u}\cdot\mathbf{u}' & \mathbf{u}\cdot\mathbf{v}'\\ \mathbf{u}'\cdot\mathbf{v} & \mathbf{v}\cdot\mathbf{v}'\end{vmatrix}
\end{aligned}$$