Exercise 40

Find the distance to the point $(6, 1, 0)$ from the plane through the origin that is perpendicular to $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Solution

The equation for a plane is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0,$$

where $\mathbf{n}$ is a normal vector and $\mathbf{r}_0$ is the position vector for any point lying in the plane. The normal vector is $(1, -2, 1)$, and since the plane passes through the origin, $\mathbf{r}_0 = (0, 0, 0)$.

$$(1, -2, 1) \cdot (x - 0, y - 0, z - 0) = 0$$

$$1(x - 0) - 2(y - 0) + 1(z - 0) = 0$$

$$x - 2y + z = 0$$

Now that the equation for the plane is known, we can determine the desired distance. An equation for the line with direction vector $(1, -2, 1)$ that passes through $(6, 1, 0)$ is

$$\begin{aligned} \mathbf{y}(t) &= (1, -2, 1)t + (6, 1, 0) \\ &= (t, -2t, t) + (6, 1, 0) \\ &= (t + 6, -2t + 1, t). \end{aligned}$$

Substitute $x = t + 6$, $y = -2t + 1$, and $z = t$ into the equation for the plane and solve for t to find when the line intersects the plane.

$$(t + 6) - 2(-2t + 1) + (t) = 0 \quad \rightarrow \quad t = -\frac{2}{3}$$

The point at which the line intersects the plane is then

$$\mathbf{y}\left(-\frac{2}{3}\right) = \left(\frac{-2}{3} + 6, -2\frac{-2}{3} + 1, \frac{-2}{3}\right) = \left(\frac{16}{3}, \frac{7}{3}, -\frac{2}{3}\right).$$

Therefore, the perpendicular distance from $(6, 1, 0)$ to the plane is

$$d = \sqrt{\left(6 - \frac{16}{3}\right)^2 + \left(1 - \frac{7}{3}\right)^2 + \left(0 + \frac{2}{3}\right)^2} = 2\sqrt{\frac{2}{3}}.$$