

Exercise 15

Show that in spherical coordinates:

- (a) ρ is the length of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
 - (b) $\phi = \cos^{-1}(\mathbf{v} \cdot \mathbf{k}/\|\mathbf{v}\|)$, where $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
 - (c) $\theta = \cos^{-1}(\mathbf{u} \cdot \mathbf{i}/\|\mathbf{u}\|)$, where $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$.
-

Solution

The spherical coordinates of a point in space (x, y, z) are

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi,$$

where

$$\rho \geq 0, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \phi \leq \pi.$$

Part (a)

Calculate the length, or magnitude, of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$$\begin{aligned} \|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\| &= \sqrt{x^2 + y^2 + z^2} \\ &= \sqrt{(\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2 + (\rho \cos \phi)^2} \\ &= \sqrt{\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta + \rho^2 \cos^2 \phi} \\ &= \sqrt{\rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) + \rho^2 \cos^2 \phi} \\ &= \sqrt{\rho^2 \sin^2 \phi + \rho^2 \cos^2 \phi} \\ &= \sqrt{\rho^2 (\sin^2 \phi + \cos^2 \phi)} \\ &= \sqrt{\rho^2} \\ &= \rho \end{aligned}$$

Part (b)

Define \mathbf{v} as

$$\begin{aligned} \mathbf{v} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ &= \rho \sin \phi \cos \theta \mathbf{i} + \rho \sin \phi \sin \theta \mathbf{j} + \rho \cos \phi \mathbf{k}. \end{aligned}$$

Take the dot product of both sides with \mathbf{k} .

$$\begin{aligned} \mathbf{v} \cdot \mathbf{k} &= (\rho \sin \phi \cos \theta \mathbf{i} + \rho \sin \phi \sin \theta \mathbf{j} + \rho \cos \phi \mathbf{k}) \cdot \mathbf{k} \\ &= \rho \sin \phi \cos \theta (\mathbf{i} \cdot \mathbf{k}) + \rho \sin \phi \sin \theta (\mathbf{j} \cdot \mathbf{k}) + \rho \cos \phi (\mathbf{k} \cdot \mathbf{k}) \\ &= \rho \sin \phi \cos \theta (0) + \rho \sin \phi \sin \theta (0) + \rho \cos \phi (1) \\ &= \rho \cos \phi \end{aligned}$$

In part (a) it was found that $\|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\| = \|\mathbf{v}\| = \rho$.

$$\mathbf{v} \cdot \mathbf{k} = \|\mathbf{v}\| \cos \phi$$

Divide both sides by $\|\mathbf{v}\|$.

$$\cos \phi = \frac{\mathbf{v} \cdot \mathbf{k}}{\|\mathbf{v}\|}$$

Therefore,

$$\phi = \cos^{-1} \left(\frac{\mathbf{v} \cdot \mathbf{k}}{\|\mathbf{v}\|} \right).$$

Part (c)

Define \mathbf{u} as

$$\begin{aligned} \mathbf{u} &= x\mathbf{i} + y\mathbf{j} \\ &= \rho \sin \phi \cos \theta \mathbf{i} + \rho \sin \phi \sin \theta \mathbf{j}. \end{aligned}$$

Take the dot product of both sides with \mathbf{i} .

$$\begin{aligned} \mathbf{u} \cdot \mathbf{i} &= (\rho \sin \phi \cos \theta \mathbf{i} + \rho \sin \phi \sin \theta \mathbf{j}) \cdot \mathbf{i} \\ &= \rho \sin \phi \cos \theta (\mathbf{i} \cdot \mathbf{i}) + \rho \sin \phi \sin \theta (\mathbf{j} \cdot \mathbf{i}) \\ &= \rho \sin \phi \cos \theta (1) + \rho \sin \phi \sin \theta (0) \\ &= \rho \sin \phi \cos \theta \end{aligned} \tag{1}$$

Note that the length, or magnitude, of \mathbf{u} is

$$\begin{aligned} \|\mathbf{u}\| &= \sqrt{(\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2} \\ &= \sqrt{\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta} \\ &= \sqrt{\rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta)} \\ &= \sqrt{\rho^2 \sin^2 \phi} \\ &= \rho \sin \phi. \end{aligned}$$

As a result, equation (1) becomes

$$\mathbf{u} \cdot \mathbf{i} = \|\mathbf{u}\| \cos \theta.$$

Divide both sides by $\|\mathbf{u}\|$.

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{i}}{\|\mathbf{u}\|}$$

Therefore,

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{i}}{\|\mathbf{u}\|} \right).$$