Exercise 22

(a) Find all points $\mathbf{p} \in \mathbb{R}^3$ that have the same representation in both Cartesian and spherical coordinates.

(b) Find all points $\mathbf{p} \in \mathbb{R}^3$ that have the same representation in both Cartesian and cylindrical coordinates.

Solution

Part (a)

The transformation from Cartesian coordinates to spherical coordinates (ρ, θ, ϕ), ϕ being the angle from the polar axis, is

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi.$$

The points in $\mathbb{R}^3$ that have the same representation in both Cartesian and spherical coordinates satisfy $(x, y, z) = (\rho, \theta, \phi)$, that is,

$$\rho = \rho \sin\phi \cos\theta \tag{1}$$

$$\theta = \rho \sin\phi \sin\theta \tag{2}$$

$$\phi = \rho \cos\phi. \tag{3}$$

Solve the first equation for ρ.

$$\rho - \rho \sin\phi \cos\theta = 0$$

$$\rho(1 - \sin\phi \cos\theta) = 0$$

$$\rho = 0 \quad \text{or} \quad 1 - \sin\phi\cos\theta = 0$$

If $\rho = 0$, then equations (1), (2), and (3) imply that $\theta = 0$ and $\phi = 0$ as well. Therefore, the origin $(0, 0, 0)$ is a point that has the same representation in both Cartesian and spherical coordinates. Only one value of ϕ and one value of θ satisfy the second equation.

$$1 - \sin\phi\cos\theta = 0 \quad \Rightarrow \quad \begin{cases} \phi = \dfrac{\pi}{2} \\ \theta = 0 \end{cases}$$

This value of ϕ doesn't satisfy equation (3), though, so there are no other points.

Part (b)

The transformation from Cartesian coordinates to cylindrical coordinates (r, θ, z) is

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z.$$

The points in $\mathbb{R}^3$ that have the same representation in both Cartesian and spherical coordinates satisfy $(x, y, z) = (r, \theta, z)$, that is,

$$r = r\cos\theta \tag{4}$$
$$\theta = r\sin\theta \tag{5}$$
$$z = z. \tag{6}$$

Solve the first equation for r.

$$r - r\cos\theta = 0$$
$$r(1 - \cos\theta) = 0$$
$$r = 0 \quad \text{or} \quad 1 - \cos\theta = 0$$

If $r = 0$, then equation (5) implies that $\theta = 0$ as well. Therefore, every point on the z-axis $(0, 0, z)$ is a point that has the same representation in both Cartesian and cylindrical coordinates. Only one value of θ satisfies the second equation.

$$1 - \cos\theta = 0 \quad \Rightarrow \quad \theta = 0$$

$\theta = 0$ automatically satisfies equations (4) and (5); r and z remain arbitrary. Therefore, every point in the $\theta = 0$ plane illustrated below has the same representation in both Cartesian and cylindrical coordinates.