Exercise 2

In $\mathbb{R}^n$ show that

(a) $2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2 = \|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2$ (This is known as the ***parallelogram law***.)

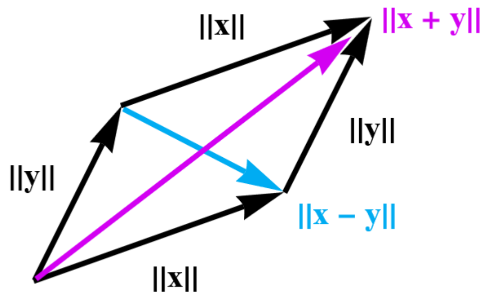
(b) $\|\mathbf{x}-\mathbf{y}\|\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$

(c) $4\langle \mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2$ (This is called the ***polarization identity***.)

Interpret these results geometrically in terms of the parallelogram formed by $\mathbf{x}$ and $\mathbf{y}$.

Solution

In a parallelogram formed by vectors, $\mathbf{x}$ and $\mathbf{y}$, $\|\mathbf{x}+\mathbf{y}\|$ and $\|\mathbf{x}-\mathbf{y}\|$ are the lengths of the diagonal lines, and $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ are the lengths of the sides.



Part (a)

Prove the first result.

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) + (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) \\
&= (\mathbf{x}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y}) + (\mathbf{x}\cdot\mathbf{x} - \mathbf{x}\cdot\mathbf{y} - \mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y}) \\
&= 2(\mathbf{x}\cdot\mathbf{x}) + 2(\mathbf{y}\cdot\mathbf{y}) \\
&= 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2
\end{aligned}$$

The parallelogram law relates the sides of a parallelogram with its diagonals.

Part (b)

Start with the square of the product of $\|\mathbf{x}+\mathbf{y}\|$ and $\|\mathbf{x}-\mathbf{y}\|$.

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2\|\mathbf{x}-\mathbf{y}\|^2 &= [(\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y})]\,[(\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y})] \\
&= (\mathbf{x}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y})\,(\mathbf{x}\cdot\mathbf{x} - \mathbf{x}\cdot\mathbf{y} - \mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y}) \\
&= \left[\|\mathbf{x}\|^2 + 2(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2\right]\left[\|\mathbf{x}\|^2 - 2(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2\right] \\
&= \|\mathbf{x}\|^4 - \cancel{2\|\mathbf{x}\|^2(\mathbf{x}\cdot\mathbf{y})} + \|\mathbf{x}\|^2\|\mathbf{y}\|^2 + \cancel{2\|\mathbf{x}\|^2(\mathbf{x}\cdot\mathbf{y})} - 4(\mathbf{x}\cdot\mathbf{y})^2 \\
&\quad + \cancel{2\|\mathbf{y}\|^2(\mathbf{x}\cdot\mathbf{y})} + \|\mathbf{x}\|^2\|\mathbf{y}\|^2 - \cancel{2\|\mathbf{y}\|^2(\mathbf{x}\cdot\mathbf{y})} + \|\mathbf{y}\|^4
\end{aligned}$$

Simplify the right side.

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2\|\mathbf{x}-\mathbf{y}\|^2 &= \|\mathbf{x}\|^4 + 2\|\mathbf{x}\|^2\|\mathbf{y}\|^2 + \|\mathbf{y}\|^4 - 4(\mathbf{x}\cdot\mathbf{y})^2 \\ &= \left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2\right)^2 - 4(\mathbf{x}\cdot\mathbf{y})^2\end{aligned}$$

Bring $4(\mathbf{x}\cdot\mathbf{y})^2$ to the left side.

$$\|\mathbf{x}+\mathbf{y}\|^2\|\mathbf{x}-\mathbf{y}\|^2 + 4(\mathbf{x}\cdot\mathbf{y})^2 = \left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2\right)^2$$

Take the square root of both sides.

$$\sqrt{\|\mathbf{x}+\mathbf{y}\|^2\|\mathbf{x}-\mathbf{y}\|^2 + 4(\mathbf{x}\cdot\mathbf{y})^2} = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

Therefore,

$$\sqrt{\|\mathbf{x}+\mathbf{y}\|^2\|\mathbf{x}-\mathbf{y}\|^2} \le \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2,$$

that is,

$$\|\mathbf{x}-\mathbf{y}\|\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2.$$

This formula indicates that the product of the diagonals is less than or equal to the sum of the squares of the parallelogram sides.

Part (c)

Prove the third result.

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) - (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) \\ &= (\mathbf{x}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y}) - (\mathbf{x}\cdot\mathbf{x} - \mathbf{x}\cdot\mathbf{y} - \mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y}) \\ &= \left[\cancel{\|\mathbf{x}\|^2} + 2(\mathbf{x}\cdot\mathbf{y}) + \cancel{\|\mathbf{y}\|^2}\right] - \left[\cancel{\|\mathbf{x}\|^2} - 2(\mathbf{x}\cdot\mathbf{y}) + \cancel{\|\mathbf{y}\|^2}\right] \\ &= 4(\mathbf{x}\cdot\mathbf{y}) \\ &= 4\langle\mathbf{x},\mathbf{y}\rangle\end{aligned}$$

This result relates the dot product of $\mathbf{x}$ and $\mathbf{y}$ with the lengths of the diagonals of the parallelogram.