Exercise 4

Verify the Cauchy–Schwarz inequality and the triangle inequality for the vectors in Exercises 3 to 6.

$$\mathbf{x} = (1, 0, 2, 6),\ \mathbf{y} = (3, 8, 4, 1)$$

Solution

Cauchy–Schwarz Inequality

Check the Cauchy–Schwarz inequality $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ for the given vectors.

$$|\mathbf{x} \cdot \mathbf{y}| = |(1)(3) + (0)(8) + (2)(4) + (6)(1)| = |17| = 17$$

$$\|\mathbf{x}\| = \sqrt{1^2 + 0^2 + 2^2 + 6^2} = \sqrt{41}$$

$$\|\mathbf{y}\| = \sqrt{3^2 + 8^2 + 4^2 + 1^2} = \sqrt{90} = 3\sqrt{10}$$

As a result,

$$|\mathbf{x} \cdot \mathbf{y}| = 17 \leq 3\sqrt{410} = \|\mathbf{x}\|\|\mathbf{y}\|,$$

which means the Cauchy–Schwarz inequality is satisfied.

Triangle Inequality

Now check the triangle inequality $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for the given vectors.

$$\mathbf{x} + \mathbf{y} = (1, 0, 2, 6) + (3, 8, 4, 1) = (4, 8, 6, 7)$$

$$\|\mathbf{x} + \mathbf{y}\| = \sqrt{4^2 + 8^2 + 6^2 + 7^2} = \sqrt{165}$$

$$\|\mathbf{x}\| = \sqrt{1^2 + 0^2 + 2^2 + 6^2} = \sqrt{41}$$

$$\|\mathbf{y}\| = \sqrt{3^2 + 8^2 + 4^2 + 1^2} = \sqrt{90} = 3\sqrt{10}$$

As a result,

$$\|\mathbf{x} + \mathbf{y}\| = \sqrt{165} \leq \sqrt{41} + 3\sqrt{10} = \|\mathbf{x}\| + \|\mathbf{y}\|,$$

which means the triangle inequality is satisfied.