Exercise 14

In Exercises 13 to 19, use set theoretic or vector notation or both to describe the points that lie in the given configurations.

The plane spanned by $\mathbf{v}_1 = (3, -1, 1)$ and $\mathbf{v}_2 = (0, 3, 4)$

Solution

These two vectors are linearly independent because one is not a constant multiple of the other. That means an entire plane is spanned by taking a linear combination of the two.

$$\begin{aligned} C_1\mathbf{v}_1 + C_2\mathbf{v}_2 &= C_1(3, -1, 1) + C_2(0, 3, 4) \\ &= (3C_1, -C_1, C_1) + (0, 3C_2, 4C_2) \\ &= (3C_1, -C_1 + 3C_2, C_1 + 4C_2) \end{aligned}$$

This plane is two-dimensional because there are two arbitrary constants, C_1 and C_2. The points in this plane are described in set notation by

$$\{(3C_1, -C_1 + 3C_2, C_1 + 4C_2),\ C_1 \in \mathbb{R},\ C_2 \in \mathbb{R}\}.$$

Any plane can be described by a vector perpendicular to it. For the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ in particular, a perpendicular vector can be obtained by taking the cross product.

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 3 & -1 & 1 \\ 0 & 3 & 4 \end{vmatrix} = -7\hat{\mathbf{x}} - 12\hat{\mathbf{y}} + 9\hat{\mathbf{z}}$$

Because this vector is normal to every vector lying in the plane, the dot product of these two is zero. The equation for a plane is obtained from this fact.

$$(\mathbf{v}_1 \times \mathbf{v}_2) \cdot (\mathbf{r} - \mathbf{v}_1) = 0$$

$$(-7, -12, 9) \cdot [(x, y, z) - (3, -1, 1)] = 0$$

$$(-7, -12, 9) \cdot (x - 3, y + 1, z - 1) = 0$$

$$-7(x - 3) - 12(y + 1) + 9(z - 1) = 0$$

Another way to describe the points in the plane using set notation is

$$\{(x, y, z) \mid -7(x - 3) - 12(y + 1) + 9(z - 1) = 0\}.$$