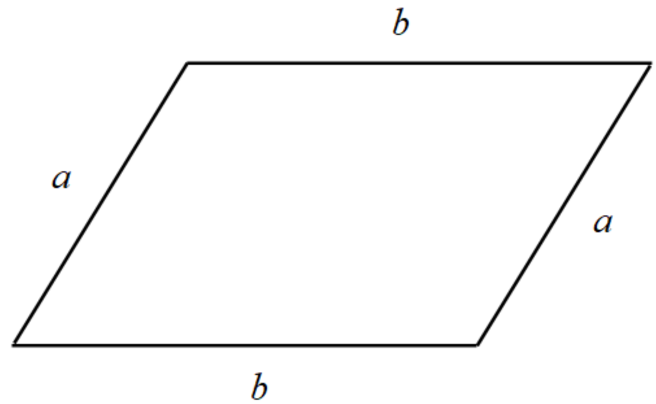
Exercise 38

Show that in any parallelogram the sum of the squares of the lengths of the four sides equals the sum of the squares of the lengths of the two diagonals.

Solution

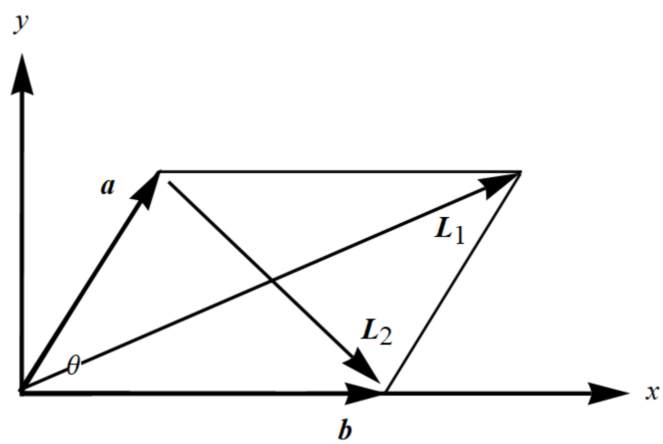
Let the adjacent sides of a parallelogram be a and b.



The sum of the squares of the lengths is then

$$\begin{aligned} a^2 + b^2 + a^2 + b^2 &= 2a^2 + b^2 \\ &= 2(a^2 + b^2) \end{aligned}$$

Now the aim is to find the length of the long and short diagonals by using vector methods.



From the figure,

$$\begin{aligned} \mathbf{a} &= (a\cos\theta, a\sin\theta) \\ \mathbf{b} &= (b, 0) \end{aligned}$$

and

$$\begin{aligned} \mathbf{L}_1 &= \mathbf{a} + \mathbf{b} = (a\cos\theta + b, a\sin\theta) \\ \mathbf{L}_2 &= -\mathbf{a} + \mathbf{b} = (-a\cos\theta + b, -a\sin\theta). \end{aligned}$$

Calculate the magnitude of $\mathbf{L}_1$ and $\mathbf{L}_2$. These are the lengths of the long and short diagonals, respectively.

$$\|\mathbf{L}_1\| = \sqrt{(a\cos\theta + b)^2 + (a\sin\theta)^2} = \sqrt{a^2 + b^2 + 2ab\cos\theta}$$
$$\|\mathbf{L}_2\| = \sqrt{(-a\cos\theta + b)^2 + (-a\sin\theta)^2} = \sqrt{a^2 + b^2 - 2ab\cos\theta}$$

Square both sides of each equation.

$$\|\mathbf{L}_1\|^2 = a^2 + b^2 + 2ab\cos\theta$$
$$\|\mathbf{L}_2\|^2 = a^2 + b^2 - 2ab\cos\theta$$

The sum of the squares of the diagonals is then

$$\begin{aligned}\|\mathbf{L}_1\|^2 + \|\mathbf{L}_2\|^2 &= a^2 + b^2 + 2ab\cos\theta + a^2 + b^2 - 2ab\cos\theta \\ &= 2a^2 + 2b^2 \\ &= 2(a^2 + b^2).\end{aligned}$$

Therefore, in any parallelogram the sum of the squares of the lengths of the four sides equals the sum of the squares of the lengths of the two diagonals.