

## Exercise 38

Given vectors  $\mathbf{a}$  and  $\mathbf{b}$ , do the equations  $\mathbf{x} \times \mathbf{a} = \mathbf{b}$  and  $\mathbf{x} \cdot \mathbf{a} = \|\mathbf{a}\|$  determine a unique vector  $\mathbf{x}$ ? Argue both geometrically and analytically.

### Solution

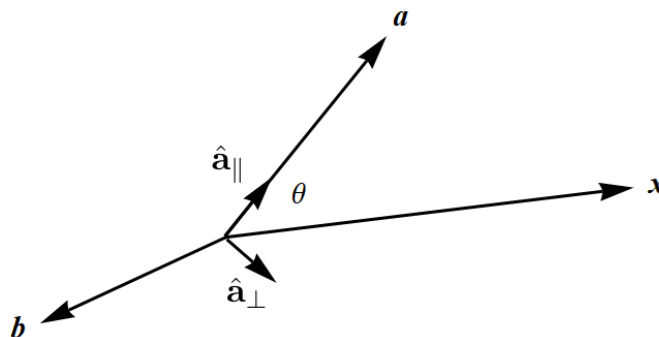
Take the magnitude of both sides of the equation involving the cross product.

$$\begin{aligned}\mathbf{x} \cdot \mathbf{a} &= \|\mathbf{a}\| \\ \|\mathbf{x} \times \mathbf{a}\| &= \|\mathbf{b}\|\end{aligned}$$

Suppose that  $\theta$  is the angle between  $\mathbf{x}$  and  $\mathbf{a}$ .

$$\begin{aligned}\|\mathbf{x}\|\|\mathbf{a}\|\cos\theta &= \|\mathbf{a}\| & \rightarrow & \|\mathbf{x}\|\cos\theta = 1 \\ \|\mathbf{x}\|\|\mathbf{a}\|\sin\theta &= \|\mathbf{b}\| & & \|\mathbf{x}\|\sin\theta = \frac{\|\mathbf{b}\|}{\|\mathbf{a}\|}\end{aligned}$$

From the equation  $\mathbf{x} \times \mathbf{a} = \mathbf{b}$ , the vector  $\mathbf{b}$  is perpendicular to the plane containing  $\mathbf{x}$  and  $\mathbf{a}$ .



A basis for this plane can be written in terms of  $\hat{\mathbf{a}}_{\parallel}$ , a unit vector parallel to  $\mathbf{a}$ , and  $\hat{\mathbf{a}}_{\perp}$ , a unit vector perpendicular to  $\mathbf{a}$  that also lies in the plane.

$$\hat{\mathbf{a}}_{\parallel} = \frac{\mathbf{a}}{\|\mathbf{a}\|} \quad \hat{\mathbf{a}}_{\perp} = \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|}$$

Write  $\mathbf{x}$  in terms of these basis vectors.

$$\mathbf{x} = x_1 \hat{\mathbf{a}}_{\parallel} + x_2 \hat{\mathbf{a}}_{\perp}$$

To determine  $x_1$ , take the dot product of both sides with  $\hat{\mathbf{a}}_{\parallel}$ .

$$\begin{aligned}\mathbf{x} \cdot \hat{\mathbf{a}}_{\parallel} &= x_1(\hat{\mathbf{a}}_{\parallel} \cdot \hat{\mathbf{a}}_{\parallel}) + x_2(\hat{\mathbf{a}}_{\perp} \cdot \hat{\mathbf{a}}_{\parallel}) \\ \|\mathbf{x}\|\cos\theta &= x_1(1) + x_2(0) \\ 1 &= x_1\end{aligned}$$

To determine  $x_2$ , take the dot product of both sides with  $\hat{\mathbf{a}}_{\perp}$ .

$$\begin{aligned}\mathbf{x} \cdot \hat{\mathbf{a}}_{\perp} &= x_1(\hat{\mathbf{a}}_{\parallel} \cdot \hat{\mathbf{a}}_{\perp}) + x_2(\hat{\mathbf{a}}_{\perp} \cdot \hat{\mathbf{a}}_{\perp}) \\ \|\mathbf{x}\|\sin\theta &= x_1(0) + x_2(1) \\ \frac{\|\mathbf{b}\|}{\|\mathbf{a}\|} &= x_2\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{x} &= x_1 \hat{\mathbf{a}}_{\parallel} + x_2 \hat{\mathbf{a}}_{\perp} \\ &= 1 \frac{\mathbf{a}}{\|\mathbf{a}\|} + \frac{\|\mathbf{b}\|}{\|\mathbf{a}\|} \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|} \\ &= \frac{\mathbf{a}}{\|\mathbf{a}\|} + \frac{\|\mathbf{b}\|}{\|\mathbf{a}\|} \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\| \sin 90^\circ} \\ &= \frac{\|\mathbf{a}\| \mathbf{a} + \mathbf{a} \times \mathbf{b}}{\|\mathbf{a}\|^2}.\end{aligned}$$