Exercise 43

Two media with indices of refraction n_1 and n_2 are separated by a plane surface perpendicular to the unit vector $\mathbf{N}$. Let $\mathbf{a}$ and $\mathbf{b}$ be unit vectors along the incident and refracted rays, respectively, their directions being those of the light rays. Show that $n_1(\mathbf{N} \times \mathbf{a}) = n_2(\mathbf{N} \times \mathbf{b})$ by using *Snell's law*, $\sin\theta_1 / \sin\theta_2 = n_2/n_1$, where θ_1 and θ_2 are the angles of incidence and refraction, respectively. (See Figure 1.3.11.)

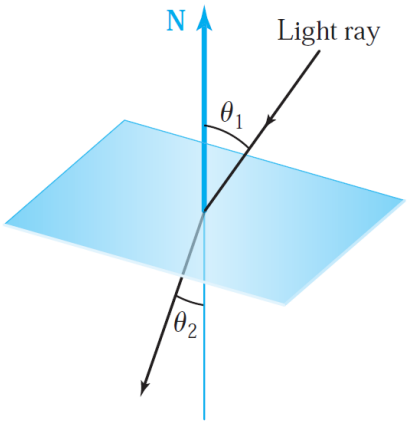


figure **1.3.11** Snell's law.

Solution

Start with Snell's law.

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1}$$

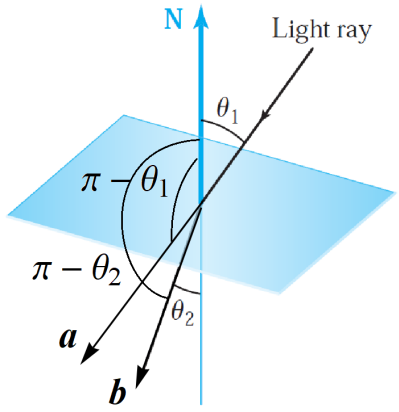
Multiply both sides by $n_1 \sin\theta_2$.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Replace the argument of each sine function by π minus the argument.

$$n_1 \sin(\pi - \theta_1) = n_2 \sin(\pi - \theta_2)$$

The point is that these new arguments represent the angles between $\mathbf{N}$ and the unit vectors as shown below.



Since $\|\mathbf{N}\| = \|\mathbf{a}\| = \|\mathbf{b}\| = 1$, they can be placed on both sides.

$$n_1\|\mathbf{N}\|\|\mathbf{a}\|\sin(\pi - \theta_1) = n_2\|\mathbf{N}\|\|\mathbf{b}\|\sin(\pi - \theta_2)$$

Use the definition of the magnitude of the cross product.

$$n_1\|\mathbf{N} \times \mathbf{a}\| = n_2\|\mathbf{N} \times \mathbf{b}\|$$

Since the incident and refracted light rays both point downward, $\mathbf{N} \times \mathbf{a}$ and $\mathbf{N} \times \mathbf{b}$ will have the same direction. Therefore,

$$n_1(\mathbf{N} \times \mathbf{a}) = n_2(\mathbf{N} \times \mathbf{b}).$$