Exercise 45

Show that adding a multiple of the first row of a matrix to the second row leaves the determinant unchanged; that is,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 + \lambda a_1 & b_2 + \lambda b_1 & c_2 + \lambda c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

[In fact, adding a multiple of any row (column) of a matrix to another row (column) leaves the determinant unchanged.]

Solution

$$\begin{aligned} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 + \lambda a_1 & b_2 + \lambda b_1 & c_2 + \lambda c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} &= (a_1, b_1, c_1) \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_2 + \lambda a_1 & b_2 + \lambda b_1 & c_2 + \lambda c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= (a_1, b_1, c_1) \cdot [(a_2 + \lambda a_1, b_2 + \lambda b_1, c_2 + \lambda c_1) \times (a_3, b_3, c_3)] \\ &= (a_1, b_1, c_1) \cdot \{[(a_2, b_2, c_2) + (\lambda a_1, \lambda b_1, \lambda c_1)] \times (a_3, b_3, c_3)\} \\ &= (a_1, b_1, c_1) \cdot \{[(a_2, b_2, c_2) + \lambda(a_1, b_1, c_1)] \times (a_3, b_3, c_3)\} \\ &= (a_1, b_1, c_1) \cdot [(a_2, b_2, c_2) \times (a_3, b_3, c_3) + \lambda(a_1, b_1, c_1) \times (a_3, b_3, c_3)] \\ &= (a_1, b_1, c_1) \cdot [(a_2, b_2, c_2) \times (a_3, b_3, c_3)] \\ &\quad + (a_1, b_1, c_1) \cdot [\lambda(a_1, b_1, c_1) \times (a_3, b_3, c_3)] \end{aligned}$$

Since $\lambda(a_1, b_1, c_1) \times (a_3, b_3, c_3)$ is perpendicular to $\lambda(a_1, b_1, c_1)$ and (a_1, b_1, c_1), $(a_1, b_1, c_1) \cdot [\lambda(a_1, b_1, c_1) \times (a_3, b_3, c_3)] = 0$, which means

$$\begin{aligned} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 + \lambda a_1 & b_2 + \lambda b_1 & c_2 + \lambda c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} &= (a_1, b_1, c_1) \cdot [(a_2, b_2, c_2) \times (a_3, b_3, c_3)] \\ &= (a_1, b_1, c_1) \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \end{aligned}$$

Therefore, adding a multiple of the first row of a matrix to the second row leaves the determinant unchanged.