Exercise 46

Suppose $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are orthogonal unit vectors. Let $\mathbf{u} = \mathbf{v} \times \mathbf{w}$. Show that $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ and $\mathbf{v} = \mathbf{w} \times \mathbf{u}$.

Solution

$$\mathbf{u} = \mathbf{v} \times \mathbf{w}$$

Take the cross product of both sides with $\mathbf{v}$ and use the first result from part (a) of Exercise 23.

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (\mathbf{v} \times \mathbf{w}) \times \mathbf{v} \\
&= (\mathbf{v} \cdot \mathbf{v})\mathbf{w} - (\mathbf{w} \cdot \mathbf{v})\mathbf{v} \\
&= (\|\mathbf{v}\|\|\mathbf{v}\| \cos 0)\, \mathbf{w} - \left(\|\mathbf{w}\|\|\mathbf{v}\| \cos \frac{\pi}{2}\right) \mathbf{v} \\
&= \|\mathbf{v}\|^2 \mathbf{w} \\
&= (1)^2 \mathbf{w} \\
&= \mathbf{w}
\end{aligned}$$

Take the cross product of $\mathbf{w}$ with both sides and use the second result from part (a) of Exercise 23.

$$\begin{aligned}
\mathbf{w} \times \mathbf{u} &= \mathbf{w} \times (\mathbf{v} \times \mathbf{w}) \\
&= (\mathbf{w} \cdot \mathbf{w})\mathbf{v} - (\mathbf{w} \cdot \mathbf{v})\mathbf{w} \\
&= (\|\mathbf{w}\|\|\mathbf{w}\| \cos 0)\, \mathbf{v} - \left(\|\mathbf{w}\|\|\mathbf{v}\| \cos \frac{\pi}{2}\right) \mathbf{w} \\
&= \|\mathbf{w}\|^2 \mathbf{v} \\
&= (1)^2 \mathbf{v} \\
&= \mathbf{v}
\end{aligned}$$