Exercise 6

A triangle has vertices $(0,0,0)$, $(1,1,1)$, and $(0,-2,3)$. Find its area.

Solution

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the displacement vectors from $(0,0,0)$ to $(1,1,1)$ and $(0,-2,3)$, respectively.

$$\begin{aligned}\mathbf{r}_1 &= (1,1,1) - (0,0,0) = (1,1,1)\\ \mathbf{r}_2 &= (0,-2,3) - (0,0,0) = (0,-2,3)\end{aligned}$$

Calculate the cross product of $\mathbf{r}_1$ and $\mathbf{r}_2$.

$$\begin{aligned}\mathbf{r}_1 \times \mathbf{r}_2 &= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 1 & 1 & 1\\ 0 & -2 & 3\end{vmatrix}\\ &= \begin{vmatrix}1 & 1\\ -2 & 3\end{vmatrix}\hat{\mathbf{x}} - \begin{vmatrix}1 & 1\\ 0 & 3\end{vmatrix}\hat{\mathbf{y}} + \begin{vmatrix}1 & 1\\ 0 & -2\end{vmatrix}\hat{\mathbf{z}}\\ &= (3+2)\hat{\mathbf{x}} - (3-0)\hat{\mathbf{y}} + (-2-0)\hat{\mathbf{z}}\\ &= 5\hat{\mathbf{x}} - 3\hat{\mathbf{y}} - 2\hat{\mathbf{z}}\\ &= (5,-3,-2)\end{aligned}$$

The area of the parallelogram with sides $\mathbf{r}_1$ and $\mathbf{r}_2$ is given by the magnitude of this cross product.

$$A = \|\mathbf{r}_1 \times \mathbf{r}_2\| = \sqrt{5^2 + (-3)^2 + (-2)^2} = \sqrt{38}$$

The area of the triangle is half of this.

$$A_T = \frac{\sqrt{38}}{2}$$