Even Fibonacci numbers

Each new term in the Fibonacci sequence is generated by adding the previous two terms. By starting with 1 and 2, the first 10 terms will be:

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

By considering the terms in the Fibonacci sequence whose values do not exceed four million, find the sum of the even-valued terms.

Solution

Let a_n represent each number in the Fibonacci sequence. By definition, each term in the sequence is obtained by adding the previous two terms.

$$a_{n+2} = a_{n+1} + a_n$$

Bring all terms to the left side.

$$a_{n+2} - a_{n+1} - a_n = 0, \qquad a_0 = 1,\ a_1 = 2 \tag{1}$$

This is a homogeneous linear difference equation with constant coefficients, so it has solutions of the form, $a_n = r^n$.

$$a_n = r^n \quad \rightarrow \quad a_{n+1} = r^{n+1} = rr^n \quad \rightarrow \quad a_{n+2} = r^{n+2} = r^2 r^n$$

Substitute these formulas into the difference equation.

$$r^2 r^n - rr^n - r^n = 0$$

Divide both sides by r^n.

$$r^2 - r - 1 = 0$$

Solve for r using the quadratic formula.

$$r = \frac{1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$r = \left\{ \frac{1 - \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2} \right\}$$

Two solutions to equation (1) are $a_n = [(1 - \sqrt{5})/2]^n$ and $a_n = [(1 + \sqrt{5})/2]^n$. According to the principle of superposition, the general solution is a linear combination of these two.

$$a_n = C_1 \left(\frac{1 - \sqrt{5}}{2} \right)^n + C_2 \left(\frac{1 + \sqrt{5}}{2} \right)^n$$

Apply the two initial values to determine C_1 and C_2.

$$a_0 = C_1 + C_2 = 1$$

$$a_1 = C_1 \left(\frac{1 - \sqrt{5}}{2} \right) + C_2 \left(\frac{1 + \sqrt{5}}{2} \right) = 2$$

Solve this system of equations for C_1 and C_2.

$$C_1 = \frac{5 - 3\sqrt{5}}{10} \quad \text{and} \quad C_2 = \frac{5 + 3\sqrt{5}}{10}$$

Therefore, the nth term in the Fibonacci sequence is

$$a_n = \frac{5 - 3\sqrt{5}}{10}\left(\frac{1 - \sqrt{5}}{2}\right)^n + \frac{5 + 3\sqrt{5}}{10}\left(\frac{1 + \sqrt{5}}{2}\right)^n, \quad n = 0, 1, \ldots.$$

Evaluate the first few terms to find out what value of n results in a number over four million.

$$\begin{aligned}
a_0 &= 1 & a_{17} &= 4181 \\
a_1 &= 2 & a_{18} &= 6765 \\
a_2 &= 3 & a_{19} &= 10\,946 \\
a_3 &= 5 & a_{20} &= 17\,711 \\
a_4 &= 8 & a_{21} &= 28\,657 \\
a_5 &= 13 & a_{22} &= 46\,368 \\
a_6 &= 21 & a_{23} &= 75\,025 \\
a_7 &= 34 & a_{24} &= 121\,393 \\
a_8 &= 55 & a_{25} &= 196\,418 \\
a_9 &= 89 & a_{26} &= 317\,811 \\
a_{10} &= 144 & a_{27} &= 514\,229 \\
a_{11} &= 233 & a_{28} &= 832\,040 \\
a_{12} &= 377 & a_{29} &= 1\,346\,269 \\
a_{13} &= 610 & a_{30} &= 2\,178\,309 \\
a_{14} &= 987 & a_{31} &= 3\,524\,578 \\
a_{15} &= 1597 & a_{32} &= 5\,702\,887 \\
a_{16} &= 2584 & &
\end{aligned}$$

The highlighted terms are the even ones that need to be added. There are few enough that it can be done manually: $S = 4\,613\,732$. Formally this is done as follows.

$$S = \sum_{n=1,4,\ldots,31} a_n$$

Let $n = 3i + 1$.

$$\begin{aligned}
S &= \sum_{3i+1=1,4,\ldots,31} a_{3i+1} \\
&= \sum_{3i=0,3,\ldots,30} a_{3i+1} \\
&= \sum_{i=0}^{10} a_{3i+1}
\end{aligned}$$

Substitute the formula for a_{3i+1} and simplify.

$$\begin{aligned}
S = \sum_{i=0}^{10} a_{3i+1} &= \sum_{i=0}^{10} \left[\frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^{3i+1} + \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^{3i+1}\right] \\
&= \sum_{i=0}^{10} \left[\frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{3i} + \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{3i}\right] \\
&= \sum_{i=0}^{10} \left[\frac{20-8\sqrt{5}}{20}\left(\frac{1-\sqrt{5}}{2}\right)^{3i} + \frac{20+8\sqrt{5}}{20}\left(\frac{1+\sqrt{5}}{2}\right)^{3i}\right] \\
&= \frac{20-8\sqrt{5}}{20}\sum_{i=0}^{10}\left(\frac{1-\sqrt{5}}{2}\right)^{3i} + \frac{20+8\sqrt{5}}{20}\sum_{i=0}^{10}\left(\frac{1+\sqrt{5}}{2}\right)^{3i} \\
&= \frac{20-8\sqrt{5}}{20}\sum_{i=0}^{10}\left[\left(\frac{1-\sqrt{5}}{2}\right)^{3}\right]^{i} + \frac{20+8\sqrt{5}}{20}\sum_{i=0}^{10}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{3}\right]^{i} \\
&= \frac{20-8\sqrt{5}}{20}\sum_{i=0}^{10}(2-\sqrt{5})^{i} + \frac{20+8\sqrt{5}}{20}\sum_{i=0}^{10}(2+\sqrt{5})^{i}
\end{aligned}$$

Recall the known summation formula,

$$\sum_{i=0}^{N} b^i = \frac{b^{N+1}-1}{b-1}, \quad b \neq 1.$$

Therefore,

$$S = \frac{20-8\sqrt{5}}{20}\cdot\frac{(2-\sqrt{5})^{11}-1}{(2-\sqrt{5})-1} + \frac{20+8\sqrt{5}}{20}\cdot\frac{(2+\sqrt{5})^{11}-1}{(2+\sqrt{5})-1} = 4\,613\,732.$$