Question Q1.13

Can you find two vectors with different lengths that have a vector sum of zero? What length restrictions are required for three vectors to have a vector sum of zero? Explain your reasoning.

Solution

The only way to have two vectors sum to zero is if they are equal and opposite.

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$$

$$\langle v_{1x}, v_{1y}, v_{1z} \rangle + \langle v_{2x}, v_{2y}, v_{2z} \rangle = \langle 0, 0, 0 \rangle$$

$$\langle v_{1x} + v_{2x}, v_{1y} + v_{2y}, v_{1z} + v_{2z} \rangle = \langle 0, 0, 0 \rangle$$

$$\left.\begin{aligned} v_{1x} + v_{2x} &= 0 \\ v_{1y} + v_{2y} &= 0 \\ v_{1z} + v_{2z} &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} v_{2x} &= -v_{1x} \\ v_{2y} &= -v_{1y} \\ v_{2z} &= -v_{1z} \end{aligned}\right\}$$

$$\begin{cases} \mathbf{v}_1 = \langle v_{1x}, v_{1y}, v_{1z} \rangle \\ \mathbf{v}_2 = \langle -v_{1x}, -v_{1y}, -v_{1z} \rangle = -\langle v_{1x}, v_{1y}, v_{1z} \rangle = -\mathbf{v}_1 \end{cases}$$

For three vectors to sum to zero, it's necessary to have

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$$

$$\langle v_{1x}, v_{1y}, v_{1z} \rangle + \langle v_{2x}, v_{2y}, v_{2z} \rangle + \langle v_{3x}, v_{3y}, v_{3z} \rangle = \langle 0, 0, 0 \rangle$$

$$\langle v_{1x} + v_{2x} + v_{3x}, v_{1y} + v_{2y} + v_{3y}, v_{1z} + v_{2z} + v_{3z} \rangle = \langle 0, 0, 0 \rangle$$

$$\left.\begin{aligned} v_{1x} + v_{2x} + v_{3x} &= 0 \\ v_{1y} + v_{2y} + v_{3y} &= 0 \\ v_{1z} + v_{2z} + v_{3z} &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} v_{3x} &= -v_{1x} - v_{2x} \\ v_{3y} &= -v_{1y} - v_{2y} \\ v_{3z} &= -v_{1z} - v_{2z} \end{aligned}\right\}$$

The first two vectors are free to be anything,

$$\mathbf{v}_1 = \langle v_{1x}, v_{1y}, v_{1z} \rangle$$

$$\mathbf{v}_2 = \langle v_{2x}, v_{2y}, v_{2z} \rangle$$

but the third vector is constrained.

$$\begin{aligned} \mathbf{v}_3 &= \langle v_{3x}, v_{3y}, v_{3z} \rangle \\ &= \langle -v_{1x} - v_{2x}, -v_{1y} - v_{2y}, -v_{1z} - v_{2z} \rangle \\ &= -\langle v_{1x} + v_{2x}, v_{1y} + v_{2y}, v_{1z} + v_{2z} \rangle \end{aligned}$$