Problem 18

Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Solution

Prove this by using the principle of mathematical induction. Start by showing that the base case is true. If $n = 1$, then

$$1 = 1^2 = 1.$$

Now assume the inductive hypothesis, that is,

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2,$$

where k is a positive integer. The aim is to show that

$$1 + 3 + 5 + \cdots + [2(k + 1) - 1] = (k + 1)^2.$$

We have

$$\begin{aligned}
(k + 1)^2 &= k^2 + 2k + 1 \\
&= [1 + 3 + 5 + \cdots + (2k - 1)] + 2k + 1 \\
&= 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) \\
&= 1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1].
\end{aligned}$$

Therefore, by the principle of mathematical induction, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ if n is a positive integer.