Problem 21

Find all the solutions of the equation

$$1 + \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \frac{x^4}{8!} + \cdots = 0$$

[*Hint:* Consider the cases $x \geq 0$ and $x < 0$ separately.]

Solution

Following the hint, we will consider the cases where x is nonnegative and negative separately. If $x > 0$, then we're adding infinitely many positive terms on the left–there's no way that sum can equal zero. Plugging in $x = 0$ gives $1 = 0$, so for $x \geq 0$, there is no solution. If $x < 0$, then the odd powers of the series become negative.

$$1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots$$

If we write the series compactly, we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n.$$

Looking at the table of Taylor series on page 768, this looks very similar to the one for $\cos x$.

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Our objective is to make the series we have look like the one for $\cos x$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\sqrt{x})^{2n} \\ &= \cos(\sqrt{x})\end{aligned}$$

The problem has reduced to solving

$$\cos \sqrt{x} = 0,$$

where $x > 0$.

$$\sqrt{x} = \frac{1}{2}(2n+1)\pi, \quad n = 0, 1, 2, \ldots$$

Squaring both sides gives x.

$$x = \frac{(2n+1)^2 \pi^2}{4}$$

Remember, though, that x has to be negative. Therefore, the following values of x satisfy the equation:

$$\left\{ x \;\middle|\; x = -\frac{(2n+1)^2 \pi^2}{4},\ n \in \mathbb{N} \right\},$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$.