Exercise 5

Evaluate the line integral, where C is the given curve.

$$\int_C (x^2y + \sin x)\,dy, \quad C \text{ is the arc of the parabola } y = x^2 \text{ from } (0,0) \text{ to } (\pi, \pi^2)$$

Solution

Parameterize the parabola by setting $x = t$, which then means $y = t^2$, and having $0 \leq t \leq \pi$. With this parameterization in t, the line integral becomes

$$\begin{aligned}
\int_C (x^2y + \sin x)\,dy &= \int_0^\pi \{[x(t)]^2 y(t) + \sin[x(t)]\} \frac{dy}{dt}\,dt \\
&= \int_0^\pi [(t)^2(t^2) + \sin(t)](2t)\,dt \\
&= 2\int_0^\pi (t^5 + t\sin t)\,dt \\
&= 2\left(\int_0^\pi t^5\,dt + \int_0^\pi t\sin t\,dt\right) \\
&= 2\left[\left(\frac{t^6}{6}\right)\bigg|_0^\pi + \int_0^\pi \frac{\partial}{\partial a}(-\cos at)\bigg|_{a=1}\,dt\right] \\
&= 2\left[\frac{\pi^6}{6} - \frac{d}{da}\left(\int_0^\pi \cos at\,dt\right)\bigg|_{a=1}\right] \\
&= 2\left[\frac{\pi^6}{6} - \frac{d}{da}\left(\frac{1}{a}\sin at\bigg|_0^\pi\right)\bigg|_{a=1}\right] \\
&= 2\left[\frac{\pi^6}{6} - \frac{d}{da}\left(\frac{\sin \pi a}{a}\right)\bigg|_{a=1}\right] \\
&= 2\left[\frac{\pi^6}{6} - \frac{(\pi\cos\pi a)a - (\sin\pi a)}{a^2}\bigg|_{a=1}\right] \\
&= 2\left[\frac{\pi^6}{6} - (-\pi)\right] \\
&= \frac{\pi^6}{3} + 2\pi.
\end{aligned}$$