Exercise 23

Solve the initial-value problem.

$$y'' - y' - 12y = 0, \quad y(1) = 0, \quad y'(1) = 1$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad \frac{dy}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y}{dx^2} = r^2e^{rx}$$

Plug these formulas into the ODE.

$$r^2e^{rx} - re^{rx} - 12(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 - r - 12 = 0$$

Solve for r.

$$(r-4)(r+3) = 0$$

$$r = \{-3, 4\}$$

Two solutions to the ODE are e^{-3x} and e^{4x}. By the principle of superposition, then,

$$y(x) = C_1e^{-3x} + C_2e^{4x}.$$

Differentiate the general solution.

$$y'(x) = -3C_1e^{-3x} + 4C_2e^{4x}$$

Apply the initial conditions to determine C_1 and C_2.

$$y(1) = C_1e^{-3} + C_2e^4 = 0$$

$$y'(1) = -3C_1e^{-3} + 4C_2e^4 = 1$$

Solving this system of equations yields $C_1 = -e^3/7$ and $C_2 = e^{-4}/7$. Therefore, the solution to the initial value problem is

$$\begin{aligned} y(x) &= -\frac{e^3}{7}e^{-3x} + \frac{e^{-4}}{7}e^{4x} \\ &= \frac{e^{4(x-1)} - e^{3(1-x)}}{7}. \end{aligned}$$

Below is a graph of $y(x)$ versus x.