Exercise 27

Solve the boundary-value problem, if possible.

$$y'' + 4y' + 4y = 0, \quad y(0) = 2, \quad y(1) = 0$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \to \quad y' = re^{rx} \quad \to \quad y'' = r^2 e^{rx}$$

Plug these formulas into the ODE.

$$r^2 e^{rx} + 4(re^{rx}) + 4(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 4r + 4 = 0$$

Solve for r.

$$(r+2)^2 = 0$$

$$r = \{-2\}$$

Two solutions to the ODE are e^{-2x} and xe^{-2x}. By the principle of superposition, then,

$$y(x) = C_1 e^{-2x} + C_2 x e^{-2x}.$$

Apply the boundary conditions to determine C_1 and C_2.

$$y(0) = C_1 = 2$$

$$y(1) = C_1 e^{-2} + C_2 e^{-2} = 0$$

Solving this system of equations yields $C_1 = 2$ and $C_2 = -2$. Therefore, the solution to the boundary value problem is

$$\begin{aligned} y(x) &= 2e^{-2x} - 2xe^{-2x} \\ &= 2(1-x)e^{-2x}. \end{aligned}$$

Below is a graph of $y(x)$ versus x.