Exercise 9

Solve the differential equation or initial-value problem using the method of undetermined coefficients.

$$y'' - y' = xe^x, \quad y(0) = 2, \quad y'(0) = 1$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - y_c' = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$r^2 e^{rx} - re^{rx} = 0$$

Divide both sides by e^{rx}.

$$r^2 - r = 0$$

Solve for r.

$$r(r-1) = 0$$

$$r = \{0, 1\}$$

Two solutions to the ODE are $e^0 = 1$ and e^x. By the principle of superposition, then,

$$y_c(x) = C_1 + C_2 e^x.$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' - y_p' = xe^x \tag{2}$$

Since the inhomogeneous term is a polynomial of degree 1 multiplied by an exponential function, the particular solution would be $y_p = (A + Bx)e^x$. But because Ae^x is already a solution of the complementary solution, x is multiplied by this trial function: $y_p = x(A + Bx)e^x$.

$$y_p = (Ax+Bx^2)e^x \quad \rightarrow \quad y_p' = (A+2Bx)e^x + (Ax+Bx^2)e^x \quad \rightarrow \quad y_p'' = (2B+A+2Bx)e^x + (A+2Bx+Ax+Bx^2)e^x$$

Substitute these formulas into equation (2).

$$[(2B + A + 2Bx)e^x + (A + 2Bx + Ax + Bx^2)e^x] - [(A + 2Bx)e^x + (Ax + Bx^2)e^x] = xe^x$$

$$(A + 2B)e^x + (2B)xe^x = xe^x$$

Match the coefficients on both sides to get a system of equations for A and B.

$$\left.\begin{aligned} A + 2B &= 0 \\ 2B &= 1 \end{aligned}\right\}$$

Solving it yields

$$A = -1 \quad \text{and} \quad B = \frac{1}{2},$$

which means the particular solution is

$$y_p = x\left(-1 + \frac{1}{2}x\right)e^x.$$

Therefore, the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_1 + C_2e^x + x\left(-1 + \frac{1}{2}x\right)e^x, \end{aligned}$$

where C_1 and C_2 are arbitrary constants. Differentiate it with respect to x.

$$y'(x) = C_2e^x + (-1 + x)e^x + \left(-x + \frac{1}{2}x^2\right)e^x$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} y(0) &= C_1 + C_2 = 2 \\ y'(0) &= C_2 - 1 = 1 \end{aligned}$$

Solving the system yields $C_1 = 0$ and $C_2 = 2$. Therefore,

$$y(x) = 2e^x + x\left(-1 + \frac{1}{2}x\right)e^x.$$