# Exercise 4

A force of 13 N is needed to keep a spring with a 2-kg mass stretched 0.25 m beyond its natural length. The damping constant of the spring is $c = 8$.

(a) If the mass starts at the equilibrium position with a velocity of 0.5 m/s, find its position at time $t$.

(b) Graph the position function of the mass.

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**Solution**

In order to determine the spring constant, use the fact that 13 N is needed to stretch the spring 0.25 m.

$$F = k(x - x_0)$$

$$13 \text{ N} = k(0.25 \text{ m})$$

$$k = 52 \, \frac{\text{N}}{\text{m}}$$

Apply Newton's second law to obtain the equation of motion.

$$\sum F = ma$$

Use the fact that acceleration is the second derivative of position $a = d^2x/dt^2$ and the fact that the spring force $F = -kx$ and the damping force $F = -c(dx/dt)$ are the only forces acting on the mass.

$$-c\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2}$$

Bring all terms to the left side.

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $x = e^{rt}$.

$$x = e^{rt} \quad \rightarrow \quad \frac{dx}{dt} = re^{rt} \quad \rightarrow \quad \frac{d^2x}{dt^2} = r^2e^{rt}$$

Plug these formulas into equation (1).

$$m(r^2e^{rt}) + c(re^{rt}) + k(e^{rt}) = 0$$

Divide both sides by $e^{rt}$.

$$mr^2 + cr + k = 0$$

Solve for $r$.

$$r = \frac{-c \pm i\sqrt{4mk - c^2}}{2m}$$

$$r = \left\{ \frac{-c - i\sqrt{4mk - c^2}}{2m}, \frac{-c + i\sqrt{4mk - c^2}}{2m} \right\}$$

Two solutions to the ODE are

$$\exp\left(\frac{-c - i\sqrt{4mk - c^2}}{2m}t\right) \quad \text{and} \quad \exp\left(\frac{-c + i\sqrt{4mk - c^2}}{2m}t\right).$$

By the principle of superposition, then, the general solution to equation (1) is

$$\begin{aligned}
x(t) &= C_1 \exp\left(\frac{-c - i\sqrt{4mk - c^2}}{2m}t\right) + C_2 \exp\left(\frac{-c + i\sqrt{4mk - c^2}}{2m}t\right) \\
&= C_1 e^{-ct/(2m)} \exp\left(-i\frac{\sqrt{4mk - c^2}}{2m}t\right) + C_2 e^{-ct/(2m)} \exp\left(-i\frac{\sqrt{4mk - c^2}}{2m}t\right) \\
&= C_1 e^{-ct/(2m)} \left(\cos\frac{\sqrt{4mk - c^2}}{2m}t - i\sin\frac{\sqrt{4mk - c^2}}{2m}t\right) + C_2 e^{-ct/(2m)} \left(\cos\frac{\sqrt{4mk - c^2}}{2m}t + i\sin\frac{\sqrt{4mk - c^2}}{2m}t\right) \\
&= e^{-ct/(2m)} \left[(C_1 + C_2)\cos\frac{\sqrt{4mk - c^2}}{2m}t + (-iC_1 + iC_2)\sin\frac{\sqrt{4mk - c^2}}{2m}t\right] \\
&= e^{-ct/(2m)} \left(C_3\cos\frac{\sqrt{4mk - c^2}}{2m}t + C_4\sin\frac{\sqrt{4mk - c^2}}{2m}t\right),
\end{aligned}$$

where $C_3$ and $C_4$ are arbitrary constants. Differentiate it with respect to $t$ to get the velocity.

$$\begin{aligned}
\frac{dx}{dt} = -\frac{c}{2m}e^{-ct/(2m)} &\left(C_3\cos\frac{\sqrt{4mk - c^2}}{2m}t + C_4\sin\frac{\sqrt{4mk - c^2}}{2m}t\right) \\
&+ e^{-ct/(2m)}\left(-C_3\frac{\sqrt{4mk - c^2}}{2m}\sin\frac{\sqrt{4mk - c^2}}{2m}t + C_4\frac{\sqrt{4mk - c^2}}{2m}\cos\frac{\sqrt{4mk - c^2}}{2m}t\right)
\end{aligned}$$

Apply the initial conditions, $x(0) = x_0 - x_0 = 0$ and $x'(0) = 0.5$, to determine $C_3$ and $C_4$.

$$\begin{aligned}
x(0) &= C_3 = 0 \\
\frac{dx}{dt}(0) &= -\frac{c}{2m}C_3 + C_4\frac{\sqrt{4mk - c^2}}{2m} = 0.5
\end{aligned}$$

Solving this system of equations yields

$$C_3 = 0 \quad \text{and} \quad C_4 = \frac{m}{\sqrt{4mk - c^2}},$$

meaning the displacement from equilibrium is

$$x(t) = \frac{m}{\sqrt{4mk - c^2}}e^{-ct/(2m)}\sin\frac{\sqrt{4mk - c^2}}{2m}t.$$

Therefore, plugging in $m = 2$ kg and $k = 52$ N/m and $c = 8$ N $\cdot$ s/m,

$$x(t) = \frac{e^{-2t}}{2\sqrt{22}}\sin\sqrt{22}t.$$

Below is a plot of $x(t)$ versus $t$.