Exercise 9

Suppose a spring has mass m and spring constant k and let $\omega = \sqrt{k/m}$. Suppose that the damping constant is so small that the damping force is negligible. If an external force $F(t) = F_0 \cos \omega_0 t$ is applied, where $\omega_0 \neq \omega$, use the method of undetermined coefficients to show that the motion of the mass is described by Equation 6.

Solution

To obtain the equation of motion for a mass attached to a spring that experiences damping and has an external force $F(t)$ applied to it, use Newton's second law.

$$\sum F = ma$$

$$F(t) - c\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2}$$

Solve for $F(t) = F_0 \cos \omega_0 t$ and set $c = 0$, since the damping is negligible.

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos \omega_0 t$$

Divide both sides by m.

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{F_0}{m}\cos \omega_0 t$$

Let $\omega^2 = k/m$.

$$\frac{d^2x}{dt^2} + \omega^2 x = \frac{F_0}{m}\cos \omega_0 t \tag{1}$$

Since this ODE is linear, the general solution can be expressed as the sum of a complementary solution and a particular solution.

$$x = x_c + x_p$$

The complementary solution satisfies the associated homogeneous equation.

$$\frac{d^2x_c}{dt^2} + \omega^2 x_c = 0 \tag{2}$$

This is a linear homogeneous ODE, so its solutions are of the form $x = e^{rt}$.

$$x_c = e^{rt} \quad \rightarrow \quad \frac{dx_c}{dt} = re^{rt} \quad \rightarrow \quad \frac{d^2x_c}{dt^2} = r^2e^{rt}$$

Plug these formulas into equation (1).

$$r^2e^{rt} + \omega^2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + \omega^2 = 0$$

Solve for r.

$$r = \{-i\omega, i\omega\}$$

Two solutions to equation (2) are $e^{-i\omega t}$ and $e^{i\omega t}$. By the principle of superposition, then,

$$\begin{aligned}x_c(t) &= C_1 e^{-i\omega t} + C_2 e^{i\omega t} \\ &= C_1(\cos\omega t - i\sin\omega t) + C_2(\cos\omega t + i\sin\omega t) \\ &= (C_1 + C_2)\cos\omega t + (-iC_1 + iC_2)\sin\omega t \\ &= C_3\cos\omega t + C_4\sin\omega t,\end{aligned}$$

where C_3 and C_4 are arbitrary constants. On the other hand, the particular solution satisfies the original ODE.

$$\frac{d^2x_p}{dt^2} + \omega^2 x_p = \frac{F_0}{m}\cos\omega_0 t \tag{3}$$

Since the inhomogeneous term is a cosine function and there are only even derivatives on the left side, $x_p = A\cos\omega_0 t$.

$$x_p = A\cos\omega_0 t \quad \rightarrow \quad \frac{dx_p}{dt} = -A\omega_0\sin\omega_0 t \quad \rightarrow \quad \frac{d^2x_p}{dt^2} = -A\omega_0^2\cos\omega_0 t$$

Substitute these formulas into equation (3).

$$(-A\omega_0^2\cos\omega_0 t) + \omega^2(A\cos\omega_0 t) = \frac{F_0}{m}\cos\omega_0 t$$

$$A(\omega^2 - \omega_0^2)\cos\omega_0 t = \frac{F_0}{m}\cos\omega_0 t$$

Match the coefficients to get an equation involving A.

$$A(\omega^2 - \omega_0^2) = \frac{F_0}{m}$$

Solve for A.

$$A = \frac{F_0}{m(\omega^2 - \omega_0^2)}$$

Therefore, the particular solution is

$$x_p = \frac{F_0}{m(\omega^2 - \omega_0^2)}\cos\omega_0 t,$$

and the general solution to equation (1) is

$$\begin{aligned}x(t) &= x_c + x_p \\ &= C_3\cos\omega t + C_4\sin\omega t + \frac{F_0}{m(\omega^2 - \omega_0^2)}\cos\omega_0 t,\end{aligned}$$

which is Equation 6 in the textbook.