Exercise 2

Solve the differential equation.

$$y'' - 2y' + 10y = 0$$

Solution

This is a linear homogeneous ODE with constant coefficients, so it has solutions of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad y' = re^{rx} \quad \rightarrow \quad y'' = r^2 e^{rx}$$

Substitute these formulas into the ODE.

$$r^2 e^{rx} - 2(re^{rx}) + 10(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 - 2r + 10 = 0$$

Solve for r.

$$r = \frac{2 \pm \sqrt{4 - 4(1)(10)}}{2} = \frac{2 \pm \sqrt{-36}}{2} = 1 \pm 3i$$

$$r = \{1 - 3i, 1 + 3i\}$$

Two solutions to the ODE are $e^{(1-3i)x}$ and $e^{(1+3i)x}$. According to the principle of superposition, the general solution to the ODE is a linear combination of these two.

$$\begin{aligned} y(x) &= C_1 e^{(1-3i)x} + C_2 e^{(1+3i)x} \\ &= C_1 e^x e^{-3ix} + C_2 e^x e^{3ix} \\ &= e^x (C_1 e^{-3ix} + C_2 e^{3ix}) \\ &= e^x [C_1(\cos 3x - i \sin 3x) + C_2(\cos 3x + i \sin 3x)] \\ &= e^x [(C_1 + C_2) \cos 3x + (-iC_1 + iC_2) \sin 3x] \end{aligned}$$

Therefore,

$$y(x) = e^x (C_3 \cos 3x + C_4 \sin 3x),$$

where C_3 and C_4 are arbitrary constants.