Exercise 21

Prove the statement using the ε, δ definition of a limit.

$$\lim_{x\to 4} \frac{x^2-2x-8}{x-4} = 6$$

Solution

According to Definition 2, proving this limit is logically equivalent to proving that

$$\text{if} \quad |x-4| < \delta \qquad \text{then} \qquad \left|\frac{x^2-2x-8}{x-4} - 6\right| < \varepsilon$$

for all positive ε. Start by working backwards, looking for a number δ that's greater than $|x-4|$.

$$\left|\frac{x^2-2x-8}{x-4} - 6\right| < \varepsilon$$

$$\left|\frac{(x-4)(x+2)}{x-4} - 6\right| < \varepsilon$$

$$|(x+2)-6| < \varepsilon$$

$$|x-4| < \varepsilon$$

Choose $\delta = \varepsilon$. Now, assuming that $|x-4| < \delta$,

$$\begin{aligned} \left|\frac{x^2-2x-8}{x-4} - 6\right| &= \left|\frac{(x-4)(x+2)}{x-4} - 6\right| \\ &= |(x+2)-6| \\ &= |x-4| \\ &< \delta \\ &= \varepsilon. \end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x\to 4} \frac{x^2-2x-8}{x-4} = 6.$$