Exercise 29

Prove the statement using the ε, δ definition of a limit.

$$\lim_{x\to 2}(x^2 - 4x + 5) = 1$$

Solution

According to Definition 2, proving this limit is logically equivalent to proving that

$$\text{if} \quad 0 < |x-2| < \delta \qquad \text{then} \qquad |(x^2 - 4x + 5) - 1| < \varepsilon$$

for all positive ε. Start by working backwards, looking for a number δ that's greater than $|x-2|$.

$$|(x^2 - 4x + 5) - 1| < \varepsilon$$

$$|x^2 - 4x + 4| < \varepsilon$$

$$|(x-2)^2| < \varepsilon$$

$$|x-2|^2 < \varepsilon$$

$$\sqrt{|x-2|^2} < \sqrt{\varepsilon}$$

$$|x-2| < \sqrt{\varepsilon}$$

Choose $\delta = \sqrt{\varepsilon}$. Now, assuming that $|x-2| < \delta$,

$$\begin{aligned} |(x^2 - 4x + 5) - 1| &= |x^2 - 4x + 4| \\ &= |(x-2)^2| \\ &= |x-2|^2 \\ &< \delta^2 \\ &= \left(\sqrt{\varepsilon}\right)^2 \\ &= \varepsilon. \end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x\to 2}(x^2 - 4x + 5) = 1.$$