Exercise 24

Find the limit or show that it does not exist.

$$\lim_{x \to -\infty} \frac{\sqrt{1+4x^6}}{2-x^3}$$

Solution

Make the substitution, $u = -x$, so that as $x \to -\infty$, $u \to \infty$. Then multiply the numerator and denominator by the reciprocal of the highest power of u in the denominator.

$$\begin{aligned}
\lim_{x \to -\infty} \frac{\sqrt{1+4x^6}}{2-x^3} = \lim_{u \to \infty} \frac{\sqrt{1+4(-u)^6}}{2-(-u)^3} = \lim_{u \to \infty} \frac{\sqrt{1+4u^6}}{2-(-u^3)} &= \lim_{u \to \infty} \frac{\sqrt{1+4u^6}}{2+u^3} \\
&= \lim_{u \to \infty} \frac{\sqrt{1+4u^6}}{2+u^3} \cdot \frac{\frac{1}{u^3}}{\frac{1}{u^3}} \\
&= \lim_{u \to \infty} \frac{\frac{1}{u^3}\sqrt{1+4u^6}}{\frac{1}{u^3}(2+u^3)} \\
&= \lim_{u \to \infty} \frac{\sqrt{\frac{1}{u^6}(1+4u^6)}}{\frac{2}{u^3}+1} \\
&= \frac{\lim_{u \to \infty} \sqrt{\frac{1}{u^6}(1+4u^6)}}{\lim_{u \to \infty} \left(\frac{2}{u^3}+1\right)} \\
&= \frac{\lim_{u \to \infty} \sqrt{\frac{1}{u^6}+4}}{\lim_{u \to \infty} \left(\frac{2}{u^3}+1\right)} \\
&= \frac{\sqrt{\lim_{u \to \infty} \left(\frac{1}{u^6}+4\right)}}{\lim_{u \to \infty} \left(\frac{2}{u^3}+1\right)} \\
&= \frac{\sqrt{\lim_{u \to \infty} \frac{1}{u^6} + \lim_{u \to \infty} 4}}{\lim_{u \to \infty} \frac{2}{u^3} + \lim_{u \to \infty} 1} \\
&= \frac{\sqrt{0+4}}{0+1} \\
&= 2
\end{aligned}$$