Exercise 36

Find the limit or show that it does not exist.

$$\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$$

Solution

Rewrite the function.

$$\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim_{x\to\infty} \frac{e^{3x} - (e^{3x})^{-1}}{e^{3x} + (e^{3x})^{-1}}$$

Make the substitution, $u = e^{3x}$. Then as $x \to \infty$, $u \to \infty$.

$$\begin{aligned}\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} &= \lim_{u\to\infty} \frac{u - u^{-1}}{u + u^{-1}} \\ &= \lim_{u\to\infty} \frac{u - \frac{1}{u}}{u + \frac{1}{u}}\end{aligned}$$

Multiply the numerator and denominator by the reciprocal of the highest power of u in the denominator.

$$\begin{aligned}\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} &= \lim_{u\to\infty} \frac{u - \frac{1}{u}}{u + \frac{1}{u}} \cdot \frac{\frac{1}{u}}{\frac{1}{u}} \\ &= \lim_{u\to\infty} \frac{\left(u - \frac{1}{u}\right)\frac{1}{u}}{\left(u + \frac{1}{u}\right)\frac{1}{u}} \\ &= \lim_{u\to\infty} \frac{1 - \frac{1}{u^2}}{1 + \frac{1}{u^2}} \\ &= \frac{\lim\limits_{u\to\infty}\left(1 - \dfrac{1}{u^2}\right)}{\lim\limits_{u\to\infty}\left(1 + \dfrac{1}{u^2}\right)} \\ &= \frac{\lim\limits_{u\to\infty} 1 - \lim\limits_{u\to\infty} \dfrac{1}{u^2}}{\lim\limits_{u\to\infty} 1 + \lim\limits_{u\to\infty} \dfrac{1}{u^2}} \\ &= \frac{1-0}{1+0} \\ &= 1\end{aligned}$$

Alternatively, without a substitution,

$$\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} \cdot \frac{e^{-3x}}{e^{-3x}} = \lim_{x\to\infty} \frac{1 - e^{-6x}}{1 + e^{-6x}} = \frac{1-0}{1+0} = 1.$$