Exercise 51

Find the horizontal and vertical asymptotes of each curve. If you have a graphing device, check your work by graphing the curve and estimating the asymptotes.

$$y = \frac{x^3 - x}{x^2 - 6x + 5}$$

Solution

Calculate the limits as $x \to \pm\infty$ to determine the horizontal asymptote. In the second limit, make the substitution, $x = -u$, so that as $x \to -\infty$, $u \to \infty$.

$$\lim_{x\to\infty} \frac{x^3 - x}{x^2 - 6x + 5} = \lim_{x\to\infty} \frac{x - \frac{1}{x}}{1 - \frac{6}{x} + \frac{5}{x^2}} = \frac{\lim_{x\to\infty} x - 0}{1 - 0} = \frac{\infty}{1} = \infty$$

$$\begin{aligned} \lim_{x\to-\infty} \frac{x^3 - x}{x^2 - 6x + 5} &= \lim_{u\to\infty} \frac{(-u)^3 - (-u)}{(-u)^2 - 6(-u) + 5} \\ &= \lim_{u\to\infty} \frac{-u^3 + u}{u^2 + 6u + 5} \\ &= \lim_{u\to\infty} \frac{-u + \frac{1}{u}}{1 + \frac{6}{u} + \frac{5}{u^2}} \\ &= \frac{\lim_{u\to\infty}(-u) + 0}{1 + 0 + 0} \\ &= \frac{-\infty}{1} = -\infty \end{aligned}$$

Since neither of these limits are finite, there are no horizontal asymptotes. Rather, there's an oblique asymptote $y = \frac{x^3}{x^2} = x$. The vertical asymptotes are found by setting what's in the denominator equal to zero and solving for x.

$$x^2 - 6x + 5 = 0$$

$$(x - 5)(x - 1) = 0$$

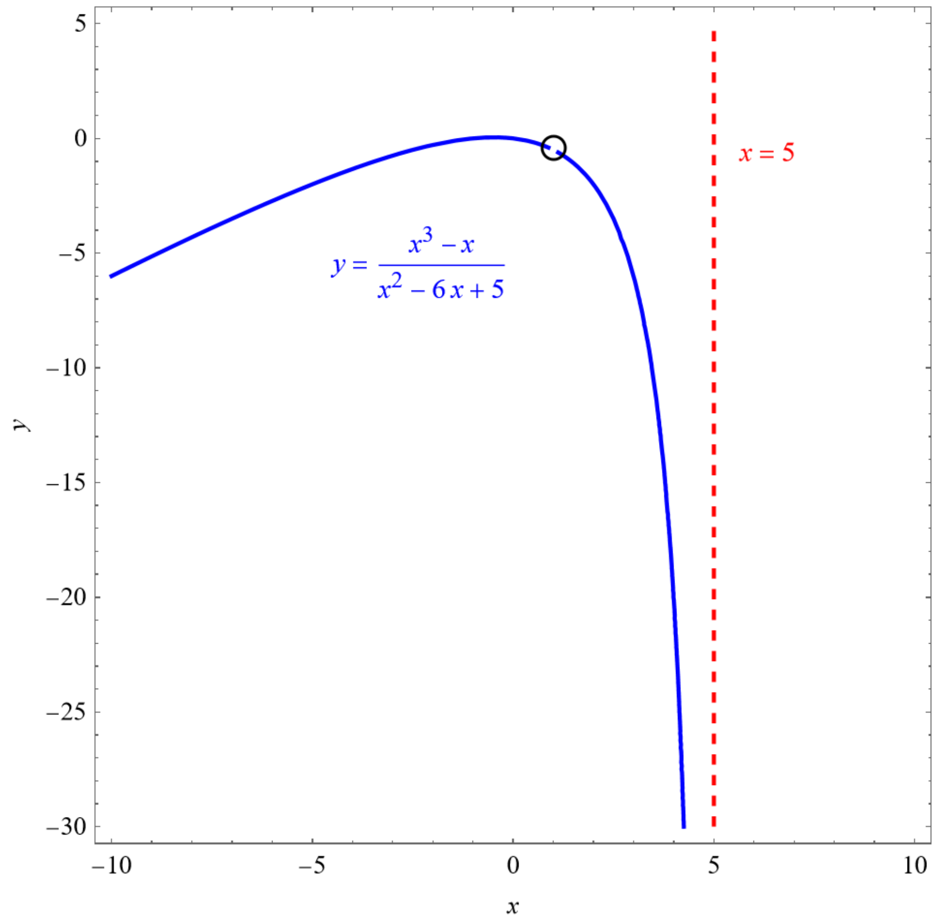
$$x = 5 \quad \text{or} \quad x = 1$$

However, because the $x - 1$ factor cancels when the function is simplified,

$$\begin{aligned} y &= \frac{x^3 - x}{x^2 - 6x + 5} \\ &= \frac{x(x^2 - 1)}{(x - 5)(x - 1)} \\ &= \frac{x(x + 1)(x - 1)}{(x - 5)(x - 1)} \\ &= \frac{x(x + 1)}{x - 5}, \end{aligned}$$

there's a hole in the graph at $x = 1$ rather than a vertical asymptote there.

The zoomed in graph below illustrates the hole at $x = 1$.



The zoomed out graph below illustrates the asymptotes.