Exercise 67

Find $\lim_{x\to\infty} f(x)$ if, for all $x > 1$,

$$\frac{10e^x - 21}{2e^x} < f(x) < \frac{5\sqrt{x}}{\sqrt{x-1}}$$

Solution

Take the limit of all sides as $x \to \infty$ and evaluate the lower and upper bounds.

$$\lim_{x\to\infty} \frac{10e^x - 21}{2e^x} < \lim_{x\to\infty} f(x) < \lim_{x\to\infty} \frac{5\sqrt{x}}{\sqrt{x-1}}$$

$$\lim_{x\to\infty} \frac{10e^x - 21}{2e^x} \cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}} < \lim_{x\to\infty} f(x) < \lim_{x\to\infty} \frac{5\sqrt{x}}{\sqrt{x-1}} \cdot \frac{\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x}}}$$

$$\lim_{x\to\infty} \frac{10 - \frac{21}{e^x}}{2} < \lim_{x\to\infty} f(x) < \lim_{x\to\infty} \frac{5}{\sqrt{1 - \frac{1}{x}}}$$

$$\frac{10-0}{2} < \lim_{x\to\infty} f(x) < \frac{5}{\sqrt{1-0}}$$

$$5 < \lim_{x\to\infty} f(x) < 5$$

Therefore,

$$\lim_{x\to\infty} f(x) = 5.$$