Exercise 63

Recall that a function f is called *even* if $f(-x) = f(x)$ for all x in its domain and *odd* if $f(-x) = -f(x)$ for all such x. Prove each of the following.

(a) The derivative of an even function is an odd function.

(b) The derivative of an odd function is an even function.

Solution

Part (a)

Suppose that $f(x)$ is an even function. Then

$$f(-x) = f(x)$$

for any x in its domain. The derivative of $f(x)$ is defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

To show that $f'(x)$ is an odd function, replace x with $-x$.

$$\begin{aligned} f'(-x) &= \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} \\ &= \lim_{h\to 0} \frac{f(-(x-h)) - f(-x)}{h} \\ &= \lim_{h\to 0} \frac{f(x-h) - f(x)}{h} \end{aligned}$$

Make the substitution, $u = -h$. As $h \to 0$, so does u.

$$\begin{aligned} f'(-x) &= \lim_{u\to 0} \frac{f(x+u) - f(x)}{-u} \\ &= -\lim_{u\to 0} \frac{f(x+u) - f(x)}{u} \\ &= -f'(x) \end{aligned}$$

Therefore, the derivative of an even function is an odd function.

Part (b)

Suppose that $f(x)$ is an odd function. Then

$$f(-x) = -f(x)$$

for any x in its domain. The derivative of $f(x)$ is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

To show that $f'(x)$ is an even function, replace x with $-x$.

$$\begin{aligned} f'(-x) &= \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} \\ &= \lim_{h \to 0} \frac{f(-(x-h)) - f(-x)}{h} \\ &= \lim_{h \to 0} \frac{-f(x-h) + f(x)}{h} \end{aligned}$$

Make the substitution, $u = -h$. As $h \to 0$, so does u.

$$\begin{aligned} f'(-x) &= \lim_{u \to 0} \frac{-f(x+u) + f(x)}{-u} \\ &= \lim_{u \to 0} \frac{f(x+u) - f(x)}{u} \\ &= f'(x) \end{aligned}$$

Therefore, the derivative of an odd function is an even function.