Exercise 64

(a) Find equations of both lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.

(b) Show that there is no line through the point $(2, 7)$ that is tangent to the parabola. Then draw a diagram to see why.

Solution

The equation of a line with slope m that passes through $(2, -3)$ is

$$y + 3 = m(x - 2).$$

Solve for y.

$$y = mx - 2m - 3$$

For this line to be tangent to the parabola, it has to intersect the parabola at exactly one point. Set the formulas equal to each other and then solve for x.

$$mx - 2m - 3 = x^2 + x$$

$$x^2 + (1 - m)x + (2m + 3) = 0$$

$$x = \frac{-(1 - m) \pm \sqrt{(1 - m)^2 - 4(2m + 3)}}{2}$$

For there to be only one intersection point, it must be that

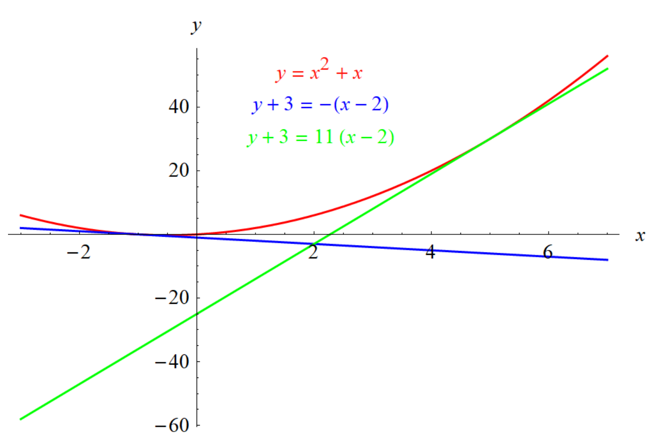
$$(1 - m)^2 - 4(2m + 3) = 0$$

$$m^2 - 10m - 11 = 0$$

$$(m + 1)(m - 11) = 0,$$

which means $m = -1$ or $m = 11$. The two tangent lines to the parabola $y = x^2 + x$ that pass through $(2, -3)$ are therefore

$$y + 3 = -(x - 2) \quad \text{and} \quad y + 3 = 11(x - 2).$$



The equation of a line with slope m that passes through $(2, 7)$ is

$$y - 7 = m(x - 2).$$

Solve for y.

$$y = mx - 2m + 7$$

For this line to be tangent to the parabola, it has to intersect the parabola at exactly one point. Set the formulas equal to each other and then solve for x.

$$mx - 2m + 7 = x^2 + x$$

$$x^2 + (1 - m)x + (2m - 7) = 0$$

$$x = \frac{-(1 - m) \pm \sqrt{(1 - m)^2 - 4(2m - 7)}}{2}$$

For there to be only one intersection point, it must be that

$$(1 - m)^2 - 4(2m - 7) = 0$$

$$m^2 - 10m + 29 = 0$$

$$m = \frac{10 \pm \sqrt{100 - 4(29)}}{2} = \frac{10 \pm \sqrt{-16}}{2} = 5 \pm 2i.$$

Because m is complex, there is no tangent line to the parabola $y = x^2 + x$ that goes through $(2, 7)$.