Exercise 52

If f is a differentiable function, find an expression for the derivative of each of the following functions.

(a) $y = x^2 f(x)$ (b) $y = \dfrac{f(x)}{x^2}$

(c) $y = \dfrac{x^2}{f(x)}$ (d) $y = \dfrac{1 + xf(x)}{\sqrt{x}}$

Solution

Use the product rule to differentiate the function in part (a).

$$y' = \frac{d}{dx}[x^2 f(x)] = \left[\frac{d}{dx}(x^2)\right] f(x) + x^2 f'(x) = (2x)f(x) + x^2 f'(x) = 2xf(x) + x^2 f'(x)$$

Use the quotient rule to differentiate the function in part (b).

$$y' = \frac{d}{dx}\left[\frac{f(x)}{x^2}\right] = \frac{f'(x)x^2 - \left[\frac{d}{dx}(x^2)\right] f(x)}{(x^2)^2} = \frac{x^2 f'(x) - 2xf(x)}{x^4}$$

Use the quotient rule to differentiate the function in part (c).

$$y' = \frac{d}{dx}\left[\frac{x^2}{f(x)}\right] = \frac{\left[\frac{d}{dx}(x^2)\right] f(x) - f'(x)(x^2)}{[f(x)]^2} = \frac{2xf(x) - x^2 f'(x)}{[f(x)]^2}$$

Use the product rule to differentiate the function in part (d).

$$\begin{aligned}
y' &= \frac{d}{dx}\left[\frac{1 + xf(x)}{\sqrt{x}}\right] \\
&= \frac{d}{dx}\left[x^{-1/2} + x^{1/2} f(x)\right] \\
&= -\frac{1}{2}x^{-3/2} + \frac{d}{dx}\left[x^{1/2} f(x)\right] \\
&= -\frac{1}{2}x^{-3/2} + \left[\frac{d}{dx}(x^{1/2})\right] f(x) + x^{1/2} f'(x) \\
&= -\frac{1}{2}x^{-3/2} + \left(\frac{1}{2}x^{-1/2}\right) f(x) + x^{1/2} f'(x) \\
&= -\frac{1}{2}x^{-3/2} + \frac{1}{2}x^{-1/2} f(x) + x^{1/2} f'(x)
\end{aligned}$$