Exercise 62

(a) If $F(x) = f(x)g(x)$, where f and g have derivatives of all orders, show that $F'' = f''g + 2f'g' + fg''$.

(b) Find similar formulas for F''' and $F^{(4)}$.

(c) Guess a formula for $F^{(n)}$.

Solution

Use the product rule to find the first derivative of $F(x)$.

$$\begin{aligned} F'(x) &= \frac{d}{dx}[f(x)g(x)] \\ &= f'(x)g(x) + f(x)g'(x) \end{aligned}$$

Use the product rule to find the second derivative of $F(x)$.

$$\begin{aligned} F''(x) &= \frac{d}{dx}[F'(x)] \\ &= \frac{d}{dx}[f'(x)g(x) + f(x)g'(x)] \\ &= \frac{d}{dx}[f'(x)g(x)] + \frac{d}{dx}[f(x)g'(x)] \\ &= \left[f''(x)g(x) + f'(x)g'(x)\right] + \left[f'(x)g'(x) + f(x)g''(x)\right] \\ &= f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x) \end{aligned}$$

Use the product rule again to find the third derivative of $F(x)$.

$$\begin{aligned} F'''(x) &= \frac{d}{dx}[F''(x)] \\ &= \frac{d}{dx}[f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)] \\ &= \frac{d}{dx}[f''(x)g(x)] + 2\frac{d}{dx}[f'(x)g'(x)] + \frac{d}{dx}[f(x)g''(x)] \\ &= \left[f'''(x)g(x) + f''(x)g'(x)\right] + 2\left[f''(x)g'(x) + f'(x)g''(x)\right] + \left[f'(x)g''(x) + f(x)g'''(x)\right] \\ &= f'''(x)g(x) + 3f''(x)g'(x) + 3f'(x)g''(x) + f(x)g'''(x) \end{aligned}$$

Use the product rule again to find the fourth derivative of $F(x)$.

$$\begin{aligned}
F''''(x) &= \frac{d}{dx}[F'''(x)] \\
&= \frac{d}{dx}[f'''(x)g(x) + 3f''(x)g'(x) + 3f'(x)g''(x) + f(x)g'''(x)] \\
&= \frac{d}{dx}[f'''(x)g(x)] + 3\frac{d}{dx}[f''(x)g'(x)] + 3\frac{d}{dx}[f'(x)g''(x)] + \frac{d}{dx}[f(x)g'''(x)] \\
&= \left[f''''(x)g(x) + f'''(x)g'(x)\right] + 3\left[f'''(x)g'(x) + f''(x)g''(x)\right] \\
&\qquad + 3\left[f''(x)g''(x) + f'(x)g'''(x)\right] + \left[f'(x)g'''(x) + f(x)g''''(x)\right] \\
&= f''''(x)g(x) + 4f'''(x)g'(x) + 6f''(x)g''(x) + 4f'(x)g'''(x) + f(x)g''''(x)
\end{aligned}$$

The nth derivative of $F(x)$ seems to be

$$\begin{aligned}
F^{(n)}(x) &= \frac{d^n}{dx^n}[F(x)] \\
&= \frac{d^n}{dx^n}[f(x)g(x)] \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)g^{(k)}(x) \\
&= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f^{(n-k)}(x)g^{(k)}(x).
\end{aligned}$$