Exercise 41

Fanciful shapes can be created by using the implicit plotting capabilities of computer algebra systems.

(a) Graph the curve with equation

$$y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$$

At how many points does this curve have horizontal tangents? Estimate the x-coordinates of these points.

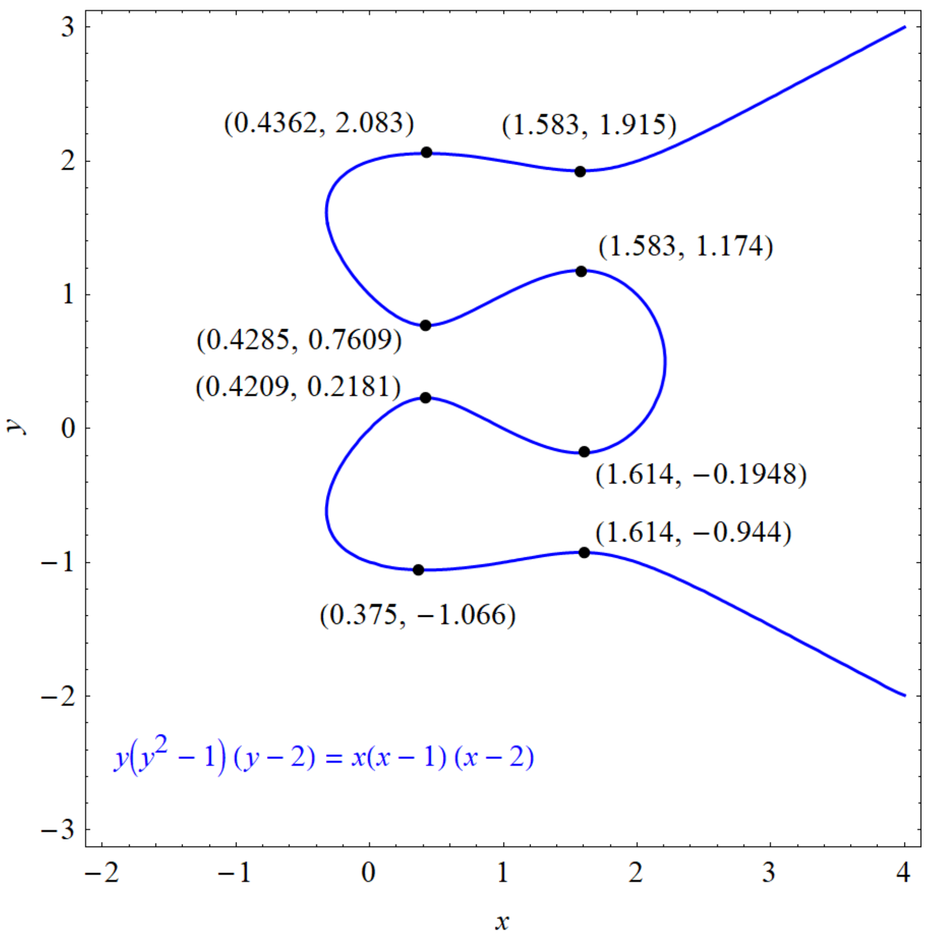
(b) Find equations of the tangent lines at the points $(0, 1)$ and $(0, 2)$.

(c) Find the exact x-coordinates of the points in part (a).

(d) Create even more fanciful curves by modifying the equation in part (a).

Solution

Below is a graph of the curve.



There are eight points where the tangent line is horizontal, and each of them are labeled.

To find the tangent lines at the points, $(0, 1)$ and $(0, 2)$, the slope of the curve needs to be known there. Use a computer to differentiate both sides with respect to x and then solve for y'.

$$\frac{d}{dx}[y(y^2-1)(y-2)] = \frac{d}{dx}[x(x-1)(x-2)]$$

$$2(2y-1)(y^2-y-1)y' = 2+3x(x-2)$$

$$y' = \frac{2+3x(x-2)}{2(2y-1)(y^2-y-1)}$$

The slopes at each of the points can be found now.

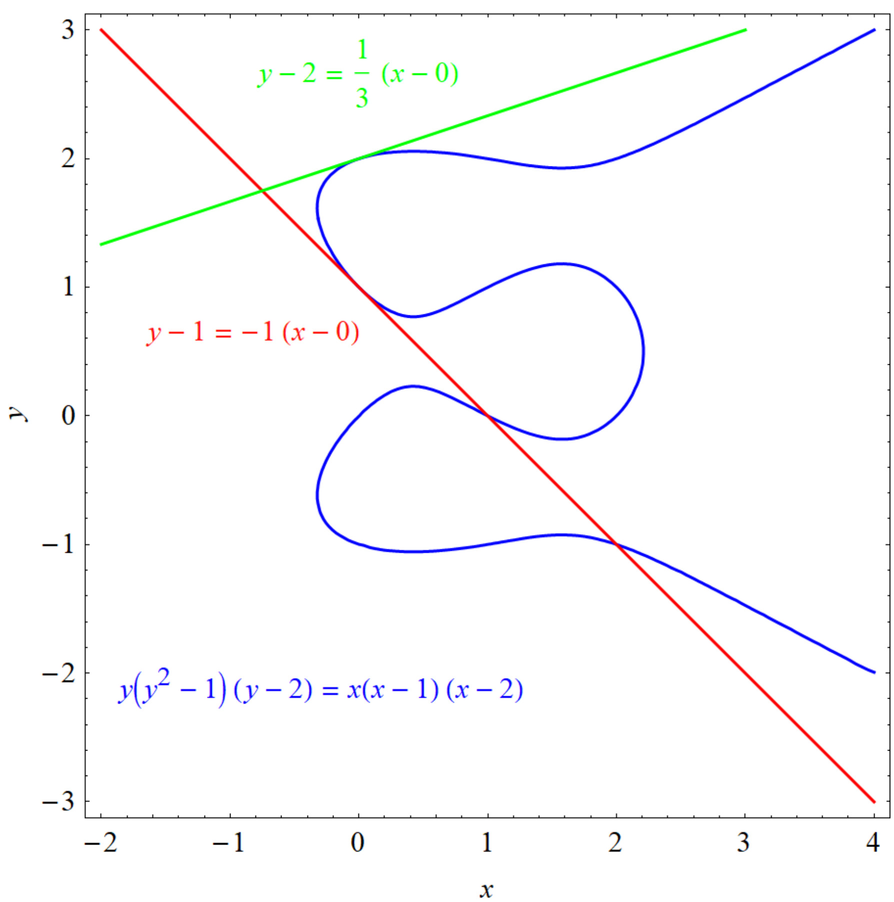
$$(0,1): \quad y'(0,1) = \frac{2+3(0)(0-2)}{2(2\cdot 1-1)(1^2-1-1)} = -1$$

$$(0,2): \quad y'(0,2) = \frac{2+3(0)(0-2)}{2(2\cdot 2-1)(2^2-2-1)} = \frac{1}{3}$$

Therefore, the tangent lines at $(0, 1)$ and $(0, 2)$ are respectively

$$y-1 = -1(x-0) \quad \text{and} \quad y-2 = \frac{1}{3}(x-0).$$

They are shown with the curve below.



To find the x-coordinates of the points where the tangent line is horizontal, set $y' = 0$ and solve for x.

$$y' = \frac{2 + 3x(x-2)}{2(2y-1)(y^2-y-1)} = 0 \quad \rightarrow \quad 2 + 3x(x-2) = 0$$

$$x = \left\{ \frac{3-\sqrt{3}}{3}, \frac{3+\sqrt{3}}{3} \right\} \approx \{0.42265, 1.57735\}$$

Adding a factor of $\sin y$ to the left side produces a more fanciful curve.