Exercise 77

(a) Suppose f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Use implicit differentiation to show that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0.

(b) If $f(4) = 5$ and $f'(4) = \frac{2}{3}$, find $(f^{-1})'(5)$.

Solution

Part (a)

Suppose that $f = f(x)$ is a one-to-one function. Then there exists an inverse function $f^{-1} = f^{-1}(x)$.

$$f[f^{-1}(x)] = x$$

Differentiate both sides with respect to x.

$$\frac{d}{dx}\{f[f^{-1}(x)]\} = \frac{d}{dx}(x)$$

Use the chain rule.

$$f'[f^{-1}(x)] \cdot \frac{d}{dx}[f^{-1}(x)] = 1$$

Therefore,

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'[f^{-1}(x)]}.$$

Part (b)

Let $f(4) = 5$ and $f'(4) = \frac{2}{3}$. Then

$$\begin{aligned} \left.\frac{d}{dx}[f^{-1}(x)]\right|_{x=5} &= \frac{1}{f'[f^{-1}(5)]} \\ &= \frac{1}{f'(4)} \\ &= \frac{1}{\frac{2}{3}} \\ &= \frac{3}{2}. \end{aligned}$$