Exercise 27

The table gives the population of the world $P(t)$, in millions, where t is measured in years and $t = 0$ corresponds to the year 1900.

t	Population (millions)	t	Population (millions)
0	1650	60	3040
10	1750	70	3710
20	1860	80	4450
30	2070	90	5280
40	2300	100	6080
50	2560	110	6870

(a) Estimate the rate of population growth in 1920 and in 1980 by averaging the slopes of two secant lines.

(b) Use a graphing device to find a cubic function (a third-degree polynomial) that models the data.

(c) Use your model in part (b) to find a model for the rate of population growth.

(d) Use part (c) to estimate the rates of growth in 1920 and 1980. Compare with your estimates in part (a).

(e) In Section 1.1 we modeled $P(t)$ with the exponential function

$$f(t) = (1.43653 \times 10^9) \cdot (1.01395)^t$$

Use this model to find a model for the rate of population growth.

(f) Use your model in part (e) to estimate the rate of growth in 1920 and 1980. Compare with your estimates in parts (a) and (d).

(g) Estimate the rate of growth in 1985.

Solution

Part (a)

Determine the slopes of the secant lines nearest $t = 20$.

$$m_{21} = \frac{P(20) - P(10)}{20 - 10} = \frac{1860 - 1750}{10} = 11\ \frac{\text{million}}{\text{year}}$$

$$m_{32} = \frac{P(30) - P(20)}{30 - 20} = \frac{2070 - 1860}{10} = 21\ \frac{\text{million}}{\text{year}}$$

Take the average of these slopes to approximate the instantaneous rate of population growth in 1920.

$$\frac{dP}{dt}(20) \approx \frac{m_{21} + m_{32}}{2} = \frac{11 + 21}{2} = 16\ \frac{\text{million}}{\text{year}}$$

Determine the slopes of the secant lines nearest $t = 80$.

$$m_{87} = \frac{P(80) - P(70)}{80 - 70} = \frac{4450 - 3710}{10} = 74\ \frac{\text{million}}{\text{year}}$$

$$m_{98} = \frac{P(90) - P(80)}{90 - 80} = \frac{5280 - 4450}{10} = 83\ \frac{\text{million}}{\text{year}}$$

Take the average of these slopes to approximate the instantaneous rate of population growth in 1980.

$$\frac{dP}{dt}(80) \approx \frac{m_{87} + m_{98}}{2} = \frac{74 + 83}{2} = 78.5\ \frac{\text{million}}{\text{year}}$$

Part (b)

Mathematica's FindFit function gives

$$P(t) \approx -0.0002849t^3 + 0.522433t^2 - 6.39564t + 1720.59$$

as the cubic function that best fits the data.

Part (c)

Take the derivative of this function to get the rate of population growth (in millions per year).

$$\begin{aligned}\frac{dP}{dt} &\approx \frac{d}{dt}(-0.0002849t^3 + 0.522433t^2 - 6.39564t + 1720.59) \\ &\approx -0.0002849(3t^2) + 0.522433(2t) - 6.39564(1) + 0 \\ &\approx -0.0008547t^2 + 1.04487t - 6.39564\end{aligned}$$

Part (d)

The instantaneous rates of population growth in 1920 and 1980 are, respectively,

$$\frac{dP}{dt}(20) \approx -0.0008547(20)^2 + 1.04487(20) - 6.39564 \approx 14.1599\ \frac{\text{million}}{\text{year}}$$

$$\frac{dP}{dt}(80) \approx -0.0008547(80)^2 + 1.04487(80) - 6.39564 \approx 71.7239\ \frac{\text{million}}{\text{year}}.$$

Use the percent difference formula to see how far the results of part (a) are from these numbers.

$$\text{1920 Percent Difference:} \quad \frac{16 - 14.1599}{14.1599} \times 100\% \approx 13\%$$

$$\text{1980 Percent Difference:} \quad \frac{78.5 - 71.7239}{71.7239} \times 100\% \approx 9.4\%$$

Therefore, the approximate results of part (a) overestimate the rates of growth in 1920 and 1980 by about 10%.

Part (e)

Differentiate the given formula for the population at time t to get the population's rate of growth.

$$\begin{aligned}
\frac{df}{dt} &= \frac{d}{dt}[(1.43653 \times 10^9) \cdot (1.01395)^t] \\
&= (1.43653 \times 10^9)\frac{d}{dt}[(1.01395)^t] \\
&= (1.43653 \times 10^9)\frac{d}{dt}[e^{\ln(1.01395)^t}] \\
&= (1.43653 \times 10^9)\frac{d}{dt}[e^{t\ln(1.01395)}] \\
&= (1.43653 \times 10^9)e^{t\ln(1.01395)} \cdot \frac{d}{dt}[t\ln(1.01395)] \\
&= (1.43653 \times 10^9)e^{t\ln(1.01395)} \cdot [\ln(1.01395)] \\
&= (1.43653 \times 10^9)\ln(1.01395)e^{t\ln(1.01395)} \\
&= (1.43653 \times 10^9)\ln(1.01395)e^{\ln(1.01395)^t} \\
&= (1.43653 \times 10^9)\ln(1.01395)(1.01395)^t
\end{aligned}$$

Part (f)

The instantaneous rates of population growth in 1920 and 1980 are, respectively,

$$\frac{df}{dt}(20) \approx (1.43653 \times 10^9)\ln(1.01395)(1.01395)^{20} \approx 26.2548\ \frac{\text{million}}{\text{year}}$$

$$\frac{df}{dt}(80) \approx (1.43653 \times 10^9)\ln(1.01395)(1.01395)^{80} \approx 60.2838\ \frac{\text{million}}{\text{year}}.$$

Use the percent difference formula to see how far the results of part (a) are from these numbers.

$$\text{1920 Percent Difference:} \quad \frac{16 - 26.2548}{26.2548} \times 100\% \approx -39\%$$

$$\text{1980 Percent Difference:} \quad \frac{78.5 - 60.2838}{60.2838} \times 100\% \approx 30\%$$

Therefore, the approximate rate of growth for 1920 underestimates the model's value by 39%, and the approximate rate of growth for 1980 overestimates the model's value by 30%.

Part (g)

According to the two population models, the rate of growth in 1985 is

$$\frac{dP}{dt}(85) \approx -0.0008547(85)^2 + 1.04487(85) - 6.39564 \approx 76.2431 \ \frac{\text{million}}{\text{year}}$$

$$\frac{df}{dt}(85) = (1.43653 \times 10^9)\ln(1.01395)(1.01395)^{85} \approx 64.6075 \ \frac{\text{million}}{\text{year}}.$$